A diffusion model in population genetics with dynamic fitness

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Abstract

We analyze a degenerate diffusion equation with singular boundary data, modeling the evolution of a polygenic trait under selection and drift. The equation models the contributions of a large but finite number of loci (genes) to the trait and at the same time allows the population trait mean to vary in a way that affects the strength of selection at individual loci; in this respect it differs from other population-genetic models that have been rigorously analyzed. We present existence, uniqueness and stability results for solutions of the system. We also prove that the genetic variance in the system tends to zero in the long time limit, and relate the dynamics of the trait mean to the variance.

1 Introduction

In population genetics one frequently encounters diffusive partial differential equations of the form

$$\phi_t = -(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx}. \tag{1}$$

Here $x \in (0,1)$ denotes the frequency of a given allele at a locus (gene), t denotes time, and $\phi(x,t)$ is a distribution indicating how many loci have allele frequency x at time t. The coefficients $M(x,t)$ and $V(x,t)$ are respectively the mean and variance of the change in allele frequency over one generation at a locus with frequency x at time t [8, Chapter 4], [5, chapter 8]. Equations such as (1) are typically posed with initial data but without boundary conditions.

Here we consider a model for a quantitative trait, i.e,. a continuous random variable whose value in an individual is generally determined by contributions from numerous loci (quantitative trait loci, or QTL) as well as non-genetic factors. (Examples of quantitative traits include the height of a human or oil content of a corn plant.) In the context of a quantitative trait, the interpretation of ϕ in equation (1) is constrained. In general, ϕ can be viewed in two ways; either as a distribution of allele frequencies at a single locus in many populations, or at many loci in one population. However, for QTL models only the latter interpretation is valid.

Our model describes the evolution of a single panmictic population of fixed size N, where n diallelic loci (we refer to the alleles as $+$ and $-$) contribute strictly additively to a single trait under selection. This is a mesoscale model in that it explicitly includes a (large) finite number of loci with finite effect, but does not retain all the information necessary to track allele frequencies at specific individual loci. Important limitations of the model are, first, that all loci are assumed to be in linkage equilibrium, and second, that mating is assumed to be random. In exchange for these simplifications, however, the model allows the population trait mean to vary in a way that affects the nature of selection on the trait. This extends previous work on diffusive PDE models of quantitative traits, which did not consider feedback of a changing trait mean on fitness functions [12].

In the classical models of population genetics [5, 14], the strength of selection at a locus is assumed to be independent of ϕ, and as a consequence, both M and V, though proportional to $x(1-x)$, are also independent of ϕ. This process yields a linear, degenerate parabolic equation. In the more flexible model we present, M depends (via a fitness function to be defined below) on the population trait mean $R(t)$, which is a nonlocal function summing contributions from all loci. We then obtain the problem

$$\phi_t = -(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx}, \tag{2a}$$
$$M = \kappa x(1-x)(\rho - R(t)), \tag{2b}$$
$$V = x(1-x), \tag{2c}$$
$$R = \left[\int_0^1 (x - \frac{1}{2})\phi(x,t)\, dx + R_0(t) + R_1(t)\right], \tag{2d}$$
$$R_0' = -\tfrac{1}{4}(V\phi)_x\big|_{x=0}, \tag{2e}$$
$$R_1' = -\tfrac{1}{4}(V\phi)_x\big|_{x=1}, \tag{2f}$$
$$\phi(\cdot,t) = \phi_0. \tag{2g}$$

Here κ represents the strength of selection, and ρ is the optimal trait value. Further, R_0 and R_1 represent contributions to the trait mean from loci at which the $+$ allele has become either fixed ($x = 1$) or lost ($x = 0$), while the integral in (2d) represents contributions from loci at which the $+$ and $-$ alleles are segregating. Because M depends on $R(t)$, we obtain a nonlinear, nonlocal equation, and because M and V remain proportional to $x(1-x)$, the problem remains degenerate. Moreover, we note that the problem as posed has no boundary conditions at $x = 0$ or $x = 1$; rather $R(t)$ and hence the coefficient M depend on the value of the boundary terms $(V\phi)_x\big|_{x=0}$ and $(V\phi)_x\big|_{x=1}$, where the equation degenerates.

Our first main result is the existence, uniqueness, and stability of solutions to a generalization to (2) where (2b) and (2c) are replaced by

$$M = x(1-x)m(x,t,R(t)) \tag{2b*}$$
$$V = x(1-x)v(x,t,R(t)) \tag{2c*}$$

with some weak hypotheses on m and v.

Our second main result is an analysis of the behavior of the trait mean $R(t)$ and (scaled) genetic variance

$$S^2(t) = \int_0^1 x(1-x)\phi(x,t)\, dx$$

as $t \to \infty$ for the original problem (2). In particular, we show that $S^2(t) \to 0$ as $t \to \infty$ in a weak sense, and that $S^2(t) = O(e^{-ct})$ for some $c > 0$ provided the initial trait mean $R(0)$ is close to the optimum trait mean ρ. We also show that

$$R(t) - \rho = (R(0) - \rho) \exp \int_0^t -\kappa\, S^2(\tau)\, d\tau.$$

implying that $R(t)$ tends monotonically towards ρ, and if the initial trait mean $R(0)$ is close to the optimum trait mean ρ that

$$|R(t) - \rho| \geq |R(0) - \rho| \exp[\gamma S^2(0)(e^{-ct} - 1)]$$

for some $c, \gamma > 0$, implying that the larger the intitial genetic variance, the closer the trait mean can come to the optimum.

In another work [16], we have used formal asymptotics to estimate the long term behavior of the population trait mean and total additive genetic variance, and have investigated numerical solutions of the system. We note that the model investigated numerically in [16] included loci of different effects, i.e. loci that varied in their contributions to the trait. For simplicity, here we assume a single effect size, which we set equal to $1/n$.

In this paper, we begin with a brief derivation of our model, including a discussion of the underlying biological assumptions. Then we present precise statements of our existence, uniqueness, and stability results, as well as how they fit into the general mathematical theory of degenerate parabolic equations. We then develop the theory of a family of weighted Sobolev spaces that are the natural spaces for energy estimates for our problem. Proofs of the existence and of the uniqueness and stability results follow. Finally, we study the long-time asymptotic behavior of the trait mean and variance.

2 The Model

As mentioned above, we consider a panmictic population of N individuals and a trait made up of strictly additive contributions from n diallelic, haploid loci (environmental contributions to the trait are ignored). The additive effect of a locus, i.e. the difference between the mean trait value of individuals carrying the $+$ allele and that of individuals carrying the $-$ allele, is taken to be a constant independent of the locus; for simplicity of notation we take that constant to be $1/n$ (we note, however, that biologically the additive effect is an important parameter and our model can readily be extended to incorporate a distribution of effect sizes; see [16]). If $x_i(t)$ denotes the percentage of individuals carrying the $+$ allele at the i-th locus in generation t, then the population trait mean is (up to an additive constant which we decree to be 0)

$$R(t) = \sum_{i=1}^{n} \left[x_i \left(\frac{1}{2n} \right) + (1 - x_i) \left(\frac{-1}{2n} \right) \right] = \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \tfrac{1}{2} \right)$$

Natural selection is modeled by a relative fitness function $f(R)$, giving the expected number of offspring of an individual whose trait value is R. We employ a Gaussian fitness function:

$$f(R) = e^{-\kappa(R-\rho)^2}$$

where ρ is the optimal trait value. Let f_{+i} and $R_{+i} = R + 1/(2n)$ (respectively, f_{-i} and $R_{-i} = R - 1/(2n)$) denote the mean fitness and the mean phenotype of individuals with a $+$ $(-)$ allele at locus i. Then the expected proportion p_i of $+$ alleles at locus i in generation $t+1$ must be proportional to both Nx_i and f_{+i}. Similar considerations apply to the expected number of $-$ alleles; it follows that

$$p_i = \frac{x_i f_{+i}}{x_i f_{+i} + (1-x_i) f_{-i}}.$$

Under weak selection and random mating, the approximation

$$p_i \approx \frac{x_i f(R_{+i})}{x_i f(R_{+i}) + (1-x_i) f(R_{-i})} \tag{3}$$

is valid [16]; we will use (3) as our definition of p_i. The actual number $Nx_i(t+1)$ of $+$ alleles at locus i in generation $t+1$ is then a binomial random variable with mean $Np_i(t)$ and variance $Np_i(t)(1-p_i(t))$.

We now introduce a probability distribution ϕ, such that $\int_a^b \phi(x,t)\,dt$ is the percentage of loci in a single population having $+$ allele frequencies between a and b at time t. A formal diffusion approximation (carried out in [16] along the lines of, e.g., [8, Chapter 4]) then yields the equation

$$\phi_t = -(M\phi)_x + \frac{1}{2}(V\phi)_{xx} \tag{4}$$

for ϕ on the x-interval $(0,1)$, with advection and diffusion coefficients M and V given by

$$M(x,t) = \frac{2\kappa x(1-x)(\rho - R(t))}{n} \tag{5}$$

$$V(x,t) = \frac{x(1-x)}{N}. \tag{6}$$

where $R(t)$ is the population trait mean. It is important to note that M and V are approximations of $p - x$ and $p(1-p)/N$ respectively, and that the resulting PDE (4) is expected to represent the corresponding discrete system exactly in the joint limit as $N \to \infty$ and $n \to \infty$, under the conditions that $n \ll N$ and that $\kappa = O(n/N)$ (so that weak selection is assumed). To complete the derivation we rescale time and the selection coefficient κ, obtaining (5) and (6).

It remains to derive an expression for $R(t)$ in terms of ϕ. Such an expression must include contributions to the trait mean from two types of loci: segregating loci (i.e. those at which $0 < x < 1$ strictly) and loci at which fixation of the $+$ $(x=1)$ or $-$ $(x=0)$ allele has occurred. To to account for these fixed loci, $R(t)$ must include terms representing mass that has passed out through the boundary of the interval $[0,1]$. Thus

$$R(t) = \int_0^1 (x - \frac{1}{2})\phi(x,t)\,dx + R_0(t) + R_1(t) \tag{7}$$

where

$$R_0(t) = -\frac{1}{4}\int_0^t (V\phi)_x(0,s)\ ds + R_0(0) \tag{8}$$

$$R_1(t) = -\frac{1}{4}\int_0^t (V\phi)_x(1,s)\ ds + R_1(0). \tag{9}$$

Equations (4)-(9), together with initial values for ϕ, R_0 and R_1, complete the specification of the model. Note that this system has no boundary conditions.

A more detailed derivation of the model is presented in [16].

3 The Main Results

Our first result is that the problem (4)-(9) has a solution, in a sense to be made precise below.

To state the result, we first need to introduce a family of Hilbert spaces. They are

$$B_0 = \left\{\psi \text{ measurable on } [0,1] : \int_0^1 x(1-x)\psi^2(x)\ dx < \infty\right\}$$

with $\langle \phi, \psi \rangle_{B_0} = \int_0^1 x(1-x)\phi\psi\ dx$,

$$B_1 = \left\{\psi \in B_0 : \int_0^1 \left[x(1-x)\psi(x)\right]_x^2\ dx < \infty\right\}$$

with $\langle \phi, \psi \rangle_{B_1} = \langle \phi, \psi \rangle_{B_0} + \int_0^1 [x(1-x)\phi]_x [x(1-x)\psi]_x\ dx$, and

$$B_2 = \left\{\psi \in B_1 : \int_0^1 x(1-x)\left[x(1-x)\psi(x)\right]_{xx}^2\ dx < \infty\right\}$$

with $\langle \phi, \psi \rangle_{B_2} = \langle \phi, \psi \rangle_{B_1} + \int_0^1 x(1-x)[x(1-x)\phi]_{xx}[x(1-x)\psi]_{xx}\ dx$.

The structure of the equation makes these the natural setting for the existence theory. Indeed, (5) and (6) imply $V = x(1-x)v$ and $M = x(1-x)m$ for functions m and v that are smooth in x with $v > 0$. Formally, if we set $m = v = 1$ to allow us to ignore lower order terms and then take the inner product of the equation with ϕ in B_0 we obtain the energy estimate

$$\frac{d}{dt}\left\|\phi(\cdot,t)\right\|_{B_0} + \left\|\phi(\cdot,t)\right\|_{B_1} \le 2\left\|\phi(\cdot,t)\right\|_{B_0}\left\|\phi(\cdot,t)\right\|_{B_1}$$

while if we multiply by $x(1-x)[x(1-x)\phi]_{xx}$ and integrate, we obtain

$$\frac{d}{dt}\left\|\phi(\cdot,t)\right\|_{B_1} + \left\|\phi(\cdot,t)\right\|_{B_2} \le 2\left\|\phi(\cdot,t)\right\|_{B_1}\left\|\phi(\cdot,t)\right\|_{B_2}.$$

Let us now state precisely our existence theorem.

Theorem 1 *Let $\phi_0 \in B_1$ be given with $\phi_0 \geq 0$, let $R_0(0)$ and $R_1(0)$ be given, and let $T > 0$. Let $M(x,t,R) = x(1-x)m(x,t,R)$ and $V(x,t,R) = x(1-x)v(x,t,R)$ satisfy the conditions:*

(H1) *The functions $(x,t,R) \mapsto m(x,t,R)$ and $(x,t,R) \mapsto v(x,r,R)$ are continuous for $0 \leq x \leq 1$, $t \geq 0$, $-\infty < R < \infty$;*

(H2) *For any $\gamma > 0$, there exist constants $C(\gamma), C'(\gamma) > 0$ so that for all $0 \leq x \leq 1$, all $t > 0$ and all $|R| \leq \gamma$*

$$
\begin{gathered}
v(x,t,R) \geq C'(\gamma), \\
|v| + |v_x| + |v_{xx}| + |m| + |m_x| \leq C(\gamma), \\
|m_R| + |v_R| + |v_{Rx}| \leq C(\gamma);
\end{gathered}
$$

(H3) *There are integrable functions $\mathcal{M}_1(t)$ and $\mathcal{M}_2(t)$ so that*

$$\sup_{0 \leq x \leq 1} |M(x,t,R)| \leq \mathcal{M}_1(t) + \mathcal{M}_2(t)|R|.$$

Then there exists a function

$$\phi \in C([0,T);B_1) \cap L_2(0,T;B_2) \cap C^{\alpha}([0,T);L_p(0,1)) \cap C((0,1) \times [0,T))$$

for any $1 \leq p < 2$, for any $0 < \alpha < \frac{1}{p} - \frac{1}{2}$; and functions

$$R_0, R_1 \in C^{\beta}[0,T);$$

for any $0 < \beta < \frac{1}{2}$ with the following properties.

Set

$$R(t) = \int_0^1 (x - \tfrac{1}{2})\phi(x,t)\,dx + R_0(t) + R_1(t).$$

Then $R \in C^1[0,T)$.

Further,

$$\phi_t = -(M(x,t,R(t))\phi)_x + \tfrac{1}{2}(V(x,t,R(t))\phi)_{xx}$$

as elements of $L_2(0,T;B_0)$ and

$$\lim_{t \downarrow 0} \phi(x,t) = \phi_0(x)$$

with the limit taken strongly in B_1.

Set

$$\nu(x,t) = \int_0^t (V(x,t,R(t))\phi)_x(x,s)\,ds$$

Then $\nu \in C^{\alpha}([0,T);C^{1-\frac{1}{p}}[0,1])$ for any $1 \leq p < 2$ and any $0 < \alpha < \frac{1}{p} - \frac{1}{2}$. Further

$$
\begin{aligned}
R_0(t) &= R_0(0) - \tfrac{1}{4}\nu(0,t), \\
R_1(t) &= R_1(0) - \tfrac{1}{4}\nu(1,t).
\end{aligned}
$$

There is a constant C depending only on T and initial data so that

$$\sup_{0\le t<T} \|\phi(\cdot,t)\|_{B_0} + \|\phi\|_{L_2(0,T;B_1)} \le C\,\|\phi_0\|_{B_0}\,,$$

$$\|\phi\|_{C^{1/2}([0,T);B_0)} \le C\,\|\phi_0\|_{B_1}\,,$$

$$\sup_{0\le t<T} \|\phi(\cdot,t)\|_{B_1} + \|\phi\|_{L_2(0,T;B_2)} \le C\,\|\phi_0\|_{B_1}\,.$$

For all $x \in (0,1)$ we have

$$\sup_{0\le t<T} |\phi(x,t)| \le C \max\left(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{1-x}}\right) \|\phi_0\|_{B_1}\,.$$

For any $1 \le p < 2$ and any $0 < \alpha < \frac{1}{p} - \frac{1}{2}$

$$\|\phi\|_{C^{\alpha}([0,T);L_p(0,1))} \le C\,\|\phi_0\|_{B_1}\,,$$

$$\|\nu\|_{C^{\alpha}([0,T);C^{1-1/p}[0,1])} \le C\,\|\phi_0\|_{B_1}\,.$$

where C also depends on p and α. Further, for any $0 < \beta < \frac{1}{2}$,

$$\|R_0\|_{C^{\beta}[0,1]} + \|R_1\|_{C^{\beta}[0,1]} \le C\,\|\phi_0\|_{B_1}\,.$$

where C also depends on β.

Moreover, $\phi \ge 0$, and for any $0 \le t_1 < t_2 < T$

$$\int_0^1 \phi(x,t_2)\,dx \le \int_0^1 \phi(x,t_1)\,dx.$$

Finally

$$|R(t)| \le \left[|R(0)| + \|\phi_0\|_{L_1(0,1)} \int_0^t \mathcal{M}_1(s)\,ds\right] \exp\left[\|\phi_0\|_{L_1(0,1)} \int_0^t \mathcal{M}_2(s)\,ds\right]$$

and

$$R(t_2) - R(t_1) = \int_{t_1}^{t_2} \int_0^1 M(x,t,R(t))\phi\,dx\,dt. \tag{10}$$

for any $0 \le t_1 < t_2 < T$.

We now briefly sketch the proof. We first develop the theory of the B_0, B_1 and B_2 spaces, which, as we remarked above, are the natural settings for our energy estimates. They also contain information about the boundary data; indeed $\psi \in B_1$ implies that $x(1-x)\psi \in \overset{\circ}{W}{}_2^1(0,1)$. Next we freeze the coefficients M and V at $\tilde{\phi} \in C([0,T);L_1)$ and $\tilde{R}_0, \tilde{R}_1 \in C[0,T)$. Application of Galerkin's method and the energy estimates allow us to uniquely solve the problem

$$\begin{cases} \phi_t = -(M\phi)_x + \frac{1}{2}(V\phi)_{xx}, \\ \phi\big|_{t=0} = \phi_0. \end{cases}$$

The full problem is nonlinear and nonlocal as the coefficients M and V depend on $R(t) = \int_0^t (x - \frac{1}{2})\phi\, dx + R_0(t) + R_1(t)$ where $R_0' = -\frac{1}{4}(V\phi)_x\big|_{x=0}$ and $R_1' = -\frac{1}{4}(V\phi)_x\big|_{x=1}$ The energy estimates suffice to control the integral, but are inadequate to control the boundary terms. In fact the energy estimates alone are insufficient to even ensure the existence of $(V\phi)_x\big|_{x=0}$ or $(V\phi)_x\big|_{x=1}$ However, the equation provides enough additional information. Indeed, the function $\nu(x,t) = \int_0^t (V\phi)_x\, ds$ satisfies the relation

$$\frac{\partial \nu}{\partial x}(x,t) = 2\int_0^t (M\phi)_x\, ds + 2[\phi(x,t) - \phi_0(x)]$$

which can formally be seen by integrating the equation. We estimate the right side and obtain enough regularity to define $\nu(0,t)$ and $\nu(1,t)$, which we then use to control R_0 and R_1.

The energy estimates depend on $\tilde{\phi}$ and $\tilde{R}_0, \tilde{R}_1$ through the quantity $\gamma = \max|\tilde{R}(t)|$. To show that the full nonlinear problem has a solution, we find a uniform bound on $\max|R(t)|$ that is independent of γ. To do so, we prove that $\int_0^1 \phi^{\pm}(x,t_2)\, dx \le \int_0^1 \phi^{\pm}(x,t_1)\, dx$ whenever $t_2 \le t_1$, which is effectively a weak maximum principle. This is done by using a regularized version of $\chi[\phi^{\pm} > 0]$ as a test function. A fixed point argument completes the proof.

We are also able to provide the following uniqueness and stability result.

Theorem 2 *Let $\phi, \phi^* \in C([0,T]; B_1) \cap L_2(0,T; B_2)$ and $R_0, R_0^*, R_1, R_1^* \in C[0,T]$. Let*

$$R(t) = \int_0^1 (x-\tfrac{1}{2})\phi(x,t) + R_0(t) + R_1(t), \quad R^*(t) = \int_0^1 (x-\tfrac{1}{2})\phi^*(x,t) + R_0^*(t) + R_1^*(t)$$

and let $M = M(x,t,R(t)$, $M^* = M(x,t,R^*(t)$, $V = V(x,t,R(t))$, and $V^* = V(x,t,R^*(t))$. Suppose that

$$\phi_t = -(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx}, \qquad \phi_t^* = -(M^*\phi^*)_x + \tfrac{1}{2}(V^*\phi^*)_{xx},$$

$$\phi\big|_{t=0} = \phi_0 \in B_1, \qquad \phi^*\big|_{t=0} = \phi_0^* \in B_1,$$

$$R(t) - R(0) = \int_0^t \int_0^1 M\phi\, dx\, dt, \qquad R^*(t) - R^*(0) = \int_0^t \int_0^1 M^*\phi^*\, dx\, dt.$$

Then

1. *If $\phi_0 = \phi_0^*$ and $R_0(0) - R_1(0) = R_0^*(0) - R_1^*(0)$ then $\phi = \phi^*$.*
2. *There is a constant C depending only on initial data and T so that*

$$\sup_{0\le t\le T} \int_0^1 x(1-x)(\phi-\phi^*)^2\, dx\Big|_t + \int_0^T \int_0^1 [x(1-x)(\phi-\phi^*)]_x^2\, dx\, dt$$
$$\le C\int_0^1 x(1-x)(\phi_0-\phi_0^*)^2\, dx + C\int_0^1 [x(1-x)(\phi_0-\phi_0^*)]_x^2 + C|R_0(0) - R_1(0) - R_0^*(0) + R_1^*(0)|^2.$$

In the uniqueness and stability result, we have encoded the conditions

$$R_0(t) = -\frac{1}{4}\int_0^t (V\phi)_x(0,s)\,ds + R_0(0) \tag{11a}$$

$$R_1(t) = -\frac{1}{4}\int_0^t (V\phi)_x(1,s)\,ds + R_1(0) \tag{11b}$$

with the requirement

$$R(t) - R(0) = \int_0^t \int_0^1 M\phi\,dx\,ds. \tag{12}$$

This is justified by (10) and Lemma 19. Formally, if we use $(x - \frac{1}{2})$ as a test function, we find

$$\int_0^t \int_0^1 \phi_t(x - \tfrac{1}{2})\,dx\,ds = -\int_0^t \int_0^1 \{(M\phi)_x(x - \tfrac{1}{2}) + \tfrac{1}{2}(V\phi)_{xx}(x - \tfrac{1}{2})\}\,dx\,ds$$

$$\int_0^1 (x - \tfrac{1}{2})\phi\,dx\Big|_0^t = \int_0^t \int_0^1 M\phi\,dx\,ds + \frac{1}{2}\int_0^t (V\phi)_x(x - \tfrac{1}{2})\,ds\Big|_{x=0}^{x=1}$$

so (formally) (11) implies (12).

To sketch the proof, we note that the structure of M and V allows us to estimate $M - M^*$ and $V - V^*$ in terms of $R - R^*$. Then (12) lets us estimate $R - R^*$ in terms of $\phi - \phi^*$ and $M - M^*$. Closing the circle gives us estimates for $M - M^*$ and $V - V^*$ in terms of $\phi - \phi^*$, which we can then use in the equation; Gronwall's inequality completes the proof.

We note that a large body of work on degenerate parabolic and elliptic systems exists, mostly based either on methods adapted from the study of nondegenerate systems or on semigroup methods (but see also [19, 20]). Classical results based on tools such as maximum principles are surveyed for nondegenerate systems in [15] and for degenerate systems in [17]. Among results in this vein, we note that Ivanov [13] proved existence in weighted Sobolev spaces to linear systems on $[0, 1]$ with a degeneracy as $x \to 0^+$ of order x^r and a similar degeneracy as $x \to 1^-$; however, these results used $r < 1$ strictly. Semigroup methods have the advantage of providing results in L^1-based spaces; for example, in [2] existence in L^1- and other L^p-based spaces is shown for solutions of the initial value problem for $u_t = (a(x)u_x)_x - b(x)u$ with a singular Neumann boundary condition. This and similar results (see for example [9]) are based in linear semigroup theory and hence are not immediately applicable to the problem studied here; it would be of interest to know whether nonlinear semigroup theory would yield analogous results.

Now we restrict our attention to our particular model; we can then prove the following result on the asymptotic behavior of the trait mean and total genetic variance.

Theorem 3 *Suppose that* $V = x(1-x)$ *and* $M = \kappa x(1-x)(\rho - R(t))$*, and let* $S^2(t) = \int_0^1 x(1-x)\phi(x,t)\,dx$*. Then*

- $S^2 \in L_1(0, \infty)$ *and*

- $R(t) - \rho = (R_0 - \rho) \exp \int_0^t -\kappa\, S^2(\tau)\, d\tau.$

Further, if there is a constant $0 < \delta < 1$ *so that* $|R_0 - \rho| \le \delta/\kappa$ *then*

- $S^2(t) \le S^2(0) e^{-(1-\delta)t}$ *and*
- $|R(t) - \rho| \ge |R_0 - \rho| \exp \left\{ S^2(0) \frac{\kappa}{1-\delta} \left(e^{-(1-\delta)t} - 1 \right) \right\}$

for any $t > 0$.

The key element in the proof is the expression (10), which in the particular case generates a linear differential equation for $R(t) - \rho$ which can be solved. Taking the inner product of the original equation with 1 in B_0 and using the particular forms for M and V, together with the expression for $R(t) - \rho$ just found, completes the proof.

We note that in what follows, the symbols N and n will be used for a variety of purposes; when this occurs it will be clear that they do not represent population size or locus number, but are simply indices.

4 Properties of B_0, B_1, and B_2

We begin the proof by collecting the necessary theory for the spaces B_0, B_1, and B_2.

Lemma 4 $C_0^\infty(0,1)$ *is dense in* B_0.

This follows immediately from the fact that if $\phi \in B_0$ then the cutoff functions $\phi\chi_{[a<x<b]}$ are in $L_2(0,1)$ for any $0 < a < b < 1$.

Lemma 5 *If* $\phi \in B_1$, *then* $x(1-x)\phi \in \overset{\circ}{W}{}_2^1(0,1)$. *Further* ϕ *has a continuous representative with* $x(1-x)\phi \in C^{1/2}[0,1]$ *so that*

$$|x_1(1-x_1)\phi(x_1) - x_2(1-x_2)\phi(x_2)| \le |x_2 - x_1|^{1/2} \left(\int_0^1 [x(1-x)\phi(x)]_x^2\, dx \right)^{1/2}.$$

Proof: Let $\phi \in B_1$; clearly $x(1-x)\phi \in W_2^1(0,1)$. We claim that for all $\epsilon > 0$ and for all $y > 0$, there exists $k > 0$ so that $k < y$ and

$$\text{meas}\{x \in (0,k) : |x(1-x)\phi(x)| \ge \epsilon\} \le \frac{1}{3}k.$$

Indeed, for any $0 < k < 1/2$ and any $\delta > 0$, we have

$$\int_0^k x(1-x)\phi^2(x)\, dx \ge \tfrac{1}{2}\delta^2 \int_0^k x\chi[|\phi| \ge \delta]\, dx.$$

If E is a measurable set, and $f(x)$ is measurable, bounded, nonnegative, and nondecreasing, them

$$\int_0^1 f(x)\chi_E\, dx \ge \int_0^{\text{meas}\, E} f(x)\, dx.$$

Indeed, this clearly holds if E is an open interval; induction then shows it holds for any countable union of open intervals and hence for any open set. Because it holds for any open set, it must hold for any measurable set. As a consequence

$$\int_0^k x(1-x)\phi^2(x)\,dx \geq \tfrac{1}{2}\delta^2 \int_0^{\mathrm{meas}_{(0,k)}[|\phi|\geq\delta]} x\,dx \geq \frac{\delta^2}{4}\left(\operatorname*{meas}_{(0,k)}[|\phi|\geq\delta]\right)^2.$$

Choose k so small that $\int_0^k x(1-x)\phi^2(x)\,dx \leq \epsilon^2/36$, and set $\delta = \epsilon/k$. Then

$$\operatorname*{meas}_{(0,k)}[|\phi| \geq \epsilon/k] \leq \frac{\epsilon}{3\epsilon/k} = \frac{k}{3}.$$

Because

$$\{x \in (0,k) : |x(1-x)\phi(x)| \geq \epsilon\} \subset \{x \in (0,k) : |\phi(x)| \geq \epsilon/k\}$$

the claim follows.

Let $\epsilon > 0$, and use the claim to choose $k \leq 1$ so that

$$k \leq \frac{\epsilon^2}{64\,\|\phi\|_{B_1}^2} \tag{13}$$

and

$$\mathrm{meas}\{x \in (0,k) : |x(1-x)\phi(x)| \geq \tfrac{\epsilon}{4}\} \leq \tfrac{k}{3}.$$

Let $A = \{x \in (0,k) : |x(1-x)\phi(x)| < \frac{\epsilon}{4}\}$; then $\mathrm{meas}\,A \geq 2k/3$. Choose $\psi \in C^1[0,1]$ so that

$$\|\psi - x(1-x)\phi\|_{W_2^1} \leq \tfrac{1}{8}\epsilon\sqrt{k}. \tag{14}$$

Let $x \in (0,k)$ and $y \in A$. Then

$$\psi(x) = \psi(y) + \int_y^x \psi'(s)\,ds$$

Integrating in y over A, and integrating in x over $(0,k)$, we find that

$$(\mathrm{meas}\,A)\int_0^k \psi(x)\,dx = k\int_A \psi(y)\,dy + \int_A\int_0^k\int_y^x \psi'(s)\,ds\,dx\,dy.$$

so that Hölder's inequality implies

$$\left|\int_0^k \psi(x)\,dx\right| \leq \frac{k}{\sqrt{\mathrm{meas}\,A}}\left(\int_A |\psi(x)|^2\,dx\right)^{1/2} + k^{3/2}\left(\int_0^k |\psi'(x)|^2\,dx\right)^{1/2}.$$

Now let $x \in (0,1)$. Then

$$\psi(0) = \psi(x) - \int_0^x \psi'(y)\,dy$$

so that if we integrate in x over $(0, k)$, we see that

$$|\psi(0)| \leq \frac{1}{k}\left|\int_0^k \psi(x)\,dx\right| + \frac{1}{k}\left|\int_0^k \int_0^x \psi'(y)\,dy\,dx\right|$$
$$\leq \frac{1}{\sqrt{\operatorname{meas} A}}\left(\int_A |\psi(x)|^2\,dx\right)^{1/2} + 2\sqrt{k}\left(\int_0^k |\psi'(x)|^2\,dx\right)^{1/2}.$$

Thus

$$|\psi(0)| \leq \frac{1}{\sqrt{\operatorname{meas} A}}\left\{\left(\int_A |x(1-x)\phi(x)|^2\,dx\right)^{1/2} + \|\psi - x(1-x)\phi\|_{L_2(0,k)}\right\}$$
$$+ 2\sqrt{k}\left\{\|[x(1-x)\phi]_x\|_{L_2(0,k)} + \|[\psi - x(1-x)\phi]_x\|_{L_2(0,k)}\right\}.$$

Now using (13), (14), and the definition of A, we determine that

$$|\psi(0)| \leq \frac{1}{\sqrt{\operatorname{meas} A}}\left\{\left(\frac{\epsilon^2}{16}\operatorname{meas} A\right)^{1/2} + \frac{\epsilon}{8}\sqrt{k}\right\} + 2\sqrt{k}\left\{\frac{\epsilon}{8\sqrt{k}} + \frac{\epsilon}{8}\sqrt{k}\right\}.$$

Then because $k < 1$ and $k \leq \frac{3}{2}\operatorname{meas} A$, we see that

$$|\psi(0)| \leq \tfrac{\epsilon}{4} + \tfrac{\epsilon}{8}\sqrt{\tfrac{3}{2}} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} < \epsilon.$$

A similar argument shows that $|\psi(1)| \leq \epsilon$.

Thus, for all $\epsilon > 0$, we can find a function $\psi \in C^1[0,1]$ so that $|\psi(0)|, |\psi(1)| < \epsilon$ and $\|\psi - x(1-x)\phi\|_{W_2^1} < \epsilon$, proving that $x(1-x) \in \overset{\circ}{W}{}_2^1(0,1)$

The rest follows from the usual Sobolev embedding results [6, IX.8]. ■

Corollary 6 *Let $\phi \in B_1$. Then*

$$\sup_{x\in[0,1]} x(1-x)\phi^2(x) \leq 2\int_0^1 [x(1-x)\phi]_x^2\,dy$$

while for any $0 < x < 1$

$$|\phi(x)| \leq 2\max\left(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{1-x}}\right)\|\phi\|_{B_1}.$$

Further, for any $1 \leq p < 2$, *there exists a constant* $C = C(p)$ *so that*

$$\|\phi\|_{L_p} \leq C\,\|\phi\|_{B_1}.$$

Proof: These follow by applying the previous with $x_1 = x$ and $x_2 = 0$ if $x \leq 1/2$ while $x_2 = 1$ if $x \geq 1/2$. ■

Lemma 7 $C_0^\infty(0,1)$ *is dense in* B_1.

Proof: Let $\phi \in B_1$, and let $\epsilon > 0$. Then $x(1-x)\phi(x) \in \overset{\circ}{W}{}_2^1(0,1)$. Choose $\tilde{\psi} \in C_0^\infty(0,1)$ so that $\left\|\tilde{\psi} - x(1-x)\phi\right\|_{W_2^1} < \frac{\epsilon}{\sqrt{3}}$ and set $\psi = \frac{1}{x(1-x)}\tilde{\psi}$. Because $\tilde{\psi} \in C_0^\infty(0,1)$, we see that $\psi \in B_1$, and hence by Corollary 6

$$\sup_{0\le x\le 1} x(1-x)[\phi-\psi]^2 \le 2\int_0^1 [x(1-x)(\phi-\psi)]_x^2\,dx.$$

Thus

$$\|\phi-\psi\|_{B_1}^2 \le 3\int_0^1 [\tilde{\psi} - x(1-x)\phi]_x^2\,dx \le \epsilon^2$$

as required. ■

Now suppose that $\phi \in B_2$. Then clearly $x(1-x)\phi(x) \in \overset{\circ}{W}{}_2^1(0,1) \cap W_{2,\text{loc}}^2(0,1)$ [6, IX.8] so that $\phi \in C_{\text{loc}}^{3/2}(0,1)$. Let

$$G(x,y) = \begin{cases} x(y-1) & x \le y \\ (x-1)y & x \ge y \end{cases}$$

be the Green's function for the problem $\psi'' = 0$, $\psi(0) = \psi(1) = 0$ [4, Chp. 7]. Then

$$\phi(x) = \frac{1}{x(1-x)}\int_0^1 G(x,y)[y(1-y)\phi]_{yy}\,dy. \tag{15}$$

Indeed, letting $I(x)$ denote the integral on the right, we see that

$$\begin{aligned} |I(x)| &\le \left(\int_0^x (1-x)y\,dy\right)^{1/2}\left(\int_0^x (1-y)y[y(1-y)\phi]_{yy}^2\,dy\right)^{1/2} \\ &\quad + \left(\int_x^1 x(1-y)\,dy\right)^{1/2}\left(\int_x^1 y(1-y)[y(1-y)\phi]_{yy}^2\,dy\right)^{1/2} \\ &\le \frac{x(1-x)}{\sqrt{2}}\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right)\left(\int_0^1 y(1-y)[y(1-y)\phi]_{yy}^2\,dy\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} |I'(x)| &\le \left(\int_0^x \frac{y}{1-y}\,dy\right)^{1/2}\left(\int_0^x (1-y)y[y(1-y)\phi]_{yy}^2\,dy\right)^{1/2} \\ &\quad + \left(\int_x^1 \frac{1-y}{y}\,dy\right)^{1/2}\left(\int_x^1 y(1-y)[y(1-y)\phi]_{yy}^2\,dy\right)^{1/2} \\ &\le \left[\left(\ln\frac{1}{x}\right)^{1/2} + \left(\ln\frac{1}{1-x}\right)^{1/2}\right]\left(\int_0^1 y(1-y)[y(1-y)\phi]_{yy}^2\,dy\right)^{1/2} \end{aligned}$$

while $I''(x) = [x(1-x)\phi]_{xx}$ for $0 < x < 1$. Since $I(0) = 0 = [x(1-x)\phi]\big|_{x=0}$ and $I(1) = 0 = [x(1-x)\phi]\big|_{x=1}$, a simple uniqueness argument implies $I(x) = x(1-x)\phi$, giving us (15).

Lemma 8 *Let $\phi \in B_2$. Then $\phi \in C^{3/2}_{loc}(0,1)$, and*

$$\int_0^1 x(1-x)\phi^2\,dx \le 2\int_0^1 x(1-x)[x(1-x)\phi]_{xx}^2\,dx$$
$$\int_0^1 [x(1-x)\phi]_x^2 \le 8\int_0^1 x(1-x)[x(1-x)\phi]_{xx}^2\,dx$$

This follows immediately from our estimates of $I(x)$ and $I'(x)$ above.

Lemma 9 $C^\infty[0,1]$ *is dense in B_2.*

Proof: Let $\phi \in B_2$, and let $\epsilon > 0$. Since $\sqrt{x(1-x)}[x(1-x)\phi]_{xx} \in L_2(0,1)$, we can choose $g \in C_0^\infty(0,1)$ so that $\left\| g - \sqrt{x(1-x)}[x(1-x)\phi]_{xx} \right\|_{L_2} < \epsilon/(1+3\sqrt{2})$. Set

$$\begin{aligned} f(x) &= \frac{1}{x(1-x)}\int_0^1 G(x,y)g(y)\,dy \\ &= \frac{1}{x}\int_0^x yg(y)\,dy + \frac{1}{1-x}\int_x^1 (1-y)g(y)\,dy. \end{aligned}$$

Clearly $f \in C^\infty[0,1]$ because g vanishes near $x = 0$ and $x = 1$; then applying Lemma 8, we see that

$$\|f-\phi\|_{B_2} \le (3\sqrt{2}+1)\left(\int_0^1 x(1-x)[x(1-x)(f-\phi)]_{xx}^2\,dx\right)^{1/2} \le \epsilon.$$

■

Remark: Because of the density of $C^\infty[0,1]$ in B_2 and of $C_0^\infty[0,1]$ in B_1 then we can justify the following integration by parts

$$\int_0^1 x(1-x)\phi[x(1-x)\psi]_{xx}\,dx = -\int_0^1 [x(1-x)\phi]_x[x(1-x)\psi]_x\,dx.$$

for any $\phi \in B_1$ and $\psi \in B_2$, which we use repeatedly in what follows.

Remark: It is easy to check that the monomials $f(x) = x^p$ are elements of B_0 if $p > -1$, elements of B_1 if $p > -1/2$, and elements of B_2 if $p > 0$. This suggests that it might be possible to generalize Lemma 5, and if $\phi \in B_2$, then $[x(1-x)\phi]_x \to 0$ as $x \downarrow 0$ or $x \uparrow 1$. However, this is not the case. Indeed, let $\zeta \in C^\infty[0,1]$ be a smooth cutoff function with $\zeta(x) = 1$ for $x \in [0, \frac{1}{3}]$, and $\zeta(x) = 0$ for $x \in [\frac{2}{3}, 1]$. Then for $0 < p < 1/2$, the function

$$f(x) = \frac{\zeta(x)}{x(1-x)}\Gamma(p+1, -\ln x).$$

is an element of B_2, but $\lim_{x\downarrow 0}[x(1-x)f(x)]_x = +\infty$. Here $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the incomplete gamma function [1, §6.5.3]

Lemma 10 *The embeddings $B_1 \hookrightarrow B_0$ and $B_1 \hookrightarrow L_p(0,1)$ for $1 \leq p < 2$ are compact.*

Proof: Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in B_1 with $\|\phi_n\|_{B_1} \leq C$. The set $\{x(1-x)\phi_n\}_{n=1}^{\infty}$ is equicontinuous and equibounded (Lemma 5), so modulo a subsequence, we see that

$$x(1-x)\phi_{n_j}(x) \longrightarrow x(1-x)\phi(x)$$

uniformly on $[0,1]$. Because B_1 is a Hilbert space, $\phi_{n_j} \rightharpoonup \phi$ weakly in B_1, modulo a second subsequence; as a consequence, $\|\phi\|_{B_1} \leq C$.

To prove that $B_1 \hookrightarrow B_0$ is compact, let $\epsilon > 0$, let $0 < \delta < \frac{\epsilon^2}{64C^2}$, and choose N so large that

$$|x(1-x)\phi_{n_j}(x) - x(1-x)\phi(x)| \leq \frac{\epsilon}{\sqrt{4\ln\left(\frac{1}{\delta}-1\right)}}$$

for all $n_j > N$ and for all x. Then

$$\left\|\phi_{n_j} - \phi\right\|_{B_0}^2 = \int_0^{\delta} x(1-x)(\phi_{n_j}-\phi)^2\,dx + \int_{\delta}^{1-\delta} x(1-x)(\phi_{n_j}-\phi)^2\,dx + \int_{1-\delta}^{1} x(1-x)(\phi_{n_j}-\phi)^2\,dx.$$

Use Corollary 6 to estimate the first and last term; then for $n_j > N$

$$\begin{aligned}\left\|\phi_{n_j} - \phi\right\|_{B_0}^2 &\leq 4\delta\left[4\left\|\phi_{n_j}\right\|_{B_1}^2 + 4\left\|\phi\right\|_{B_1}^2\right] + \int_{\delta}^{1-\delta} [x(1-x)(\phi_{n_j}-\phi)]^2 \frac{dx}{x(1-x)} \\ &\leq 32\delta C^2 + \frac{\epsilon^2}{4\ln(\frac{1}{\delta}-1)} \int_{\delta}^{1-\delta} \left(\frac{1}{x} + \frac{1}{1-x}\right) dx \leq \epsilon^2.\end{aligned}$$

Thus $\phi_n \to \phi$ in B_0.

To prove that B_1 is compact in $L_p(0,1)$, let $\epsilon > 0$, let $0 < \delta < \frac{1}{2}$ satisfy

$$\delta < \left(\frac{2-p}{8(4C)^p}\epsilon^p\right)^{2/(2-p)}$$

and for $1 < p < 2$, choose N so large that

$$|x(1-x)\phi_{n_j}(x) - x(1-x)\phi(x)| < \epsilon\left(\frac{p-1}{2^{p+2}}\delta^{p-1}\right)^{1/p} \tag{16}$$

while for $p = 1$ choose N so large that

$$|x(1-x)\phi_{n_j}(x) - x(1-x)\phi(x)| < \frac{\epsilon}{4\ln\left(\frac{1-\delta}{\delta}\right)} \tag{17}$$

for all $n_j > N$ and for all x. Corollary 6 and the definition of δ then shows

$$\int_0^{\delta} |\phi_{n_j} - \phi|^p\,dx + \int_{1-\delta}^{1} |\phi_{n_j} - \phi|^p\,dx \leq \frac{4(4C)^p}{2-p}\delta^{1-p/2} \leq \frac{\epsilon^p}{2}.$$

On the other hand if $1 < p < 2$ we can use (16) to find that

$$\int_\delta^{1-\delta} |\phi_{n_j}(x) - \phi(x)|^p \, dx \le \frac{p-1}{2^{p+2}} \delta^{p-1} \epsilon^p \int_\delta^{1-\delta} \frac{dx}{x^p(1-x)^p} \le \frac{\epsilon^p}{2}$$

while if $p = 1$ we use (17) to obtain

$$\int_\delta^{1-\delta} |\phi_{n_j}(x) - \phi(x)| \, dx \le \frac{\epsilon}{4 \ln\left(\frac{1-\delta}{\delta}\right)} \int_\delta^{1-\delta} \left(\frac{1}{x} + \frac{1}{1-x}\right) dx \le \frac{\epsilon}{2}$$

Hence $\left\|\phi_{n_j} - \phi\right\|_{L_p}^p \le \epsilon^p$ and $\phi_n \to \phi$ in $L_p(0,1)$. ■

Lemma 11 *There exists an nondecreasing sequence of nonnegative eigenvalues $\lambda_k \to \infty$ and eigenfunctions $\phi_k \in B_2$ so that $-[x(1-x)\phi_k]'' = \lambda_k \phi_k$. Further, the set $\{\phi_k\}_{k=1}^\infty$ is an orthonormal basis for B_0, and forms a basis for B_1.*

Proof (sketch): This result can be proven using standard techniques c.f. [7, §6.5] or [11, §8.12]. Indeed, for $\phi \in B_1$, consider the Rayleigh quotient

$$J(\phi) = \frac{\int_0^1 [x(1-x)\phi(x)]_x^2 \, dx}{\int_0^1 x(1-x)\phi^2 \, dx}.$$

Then $\lambda_1 = \inf\{J(\phi) : \phi \in B_1\}$. We can choose a minimizing sequence $\psi_k \in B_1$, $\|\phi_k\|_{B_0} = 1$, $J(\psi_k) \downarrow \lambda_1$ and use the compactness $B_1 \hookrightarrow B_0$ to show that there is a function ϕ_1 so that $\psi_k \to \phi_1$ in B_0 and $\psi_k \rightharpoonup \phi_1$ weakly in B_1. Then $J(\phi_1) = \lambda_1$ and $-[x(1-x)\phi_1]'' = \lambda_1 \phi_1$, so $\phi_1 \in B_2$. Subsequent eigenvectors and eigenvalues are found inductively, with $\lambda_j = \inf\{J(\phi) : \phi \in B_1, \langle \phi, \phi_m \rangle_{B_0} = 0, \text{for } 1 \le m < j\}$. If $\lambda_n \le C$ for all n, we can use the fact that $\|\phi_n\|_{B_1}^2 = 1 + \lambda_n$ to find a subsequence n_j and a function $\phi \in B_1$ with $\phi_{n_j} \to \phi$ in B_0 and $\phi_{n_j} \rightharpoonup \phi$ weakly in B_1. However $\|\phi_m - \phi_n\|_{B_0}^2 = 2$ for any eigenfunctions ϕ_m and ϕ_n, so $\lambda_n \to \infty$. Completeness in B_0 follows in the usual way. Indeed, if $\psi \in B_1$ let $\psi_n = \psi - \sum_{j=1}^n \langle \psi, \phi_j \rangle_{B_0} \phi_j$. Then $J(\psi_n) \ge \lambda_{n+1}$ so that $\lambda_{n+1} \|\psi_n\|_{B_0}^2 \le \int_0^1 [x(1-x)\psi_n]_x^2 \, dx \le \int_0^1 [x(1-x)\psi_n]_x^2 \, dx$ so that $\psi_n \to 0$ in B_0. If $\psi \in B_0 \setminus B_1$ we use the fact that $C_0^\infty[0,1] \subset B_1$ is dense in B_0. Finally, to show that the eigenfunctions are dense in B_1, it is sufficient to note that $\langle \phi, \phi_k \rangle_{B_1} = (1 + \lambda_k) \langle \phi, \phi_k \rangle_{B_0}$ for any $\phi \in B_1$ and any eigenfunction ϕ_k. If $\phi \in B_1$ was orthogonal in B_1 to every eigenfunction, then ϕ would be orthogonal to every eigenfunction in B_0 which we have already shown to be impossible. ■.

Remark: The eigenfunctions are polynomials, and the eigenvalues and eigenfunctions can be explicitly computed; in fact $\lambda_n = 2 + n(n+3)$ and the eigenfunctions ϕ_n are proportional to $C_n^{(3/2)}(2x-1)$ where $C_n^{(3/2)}(x)$ are Gegenbauer polynomials [1, §22].

5 The approximating problems

As we noted, our problem is nonlinear and nonlocal because the coefficients M and V depend depend on the solution ϕ through $R(t) = \int_0^1 (x - \frac{1}{2})\phi \, dx + R_0(t) + R_1(t)$ where $R_0' = (V\phi)_x\big|_{x=0}$ and $R_1' = (V\phi)_x\big|_{x=1}$ (in a suitable weak sense).

We begin by letting $T > 0$, and choosing

$$\tilde{\phi} \in C([0,T); L_1(0,1)),$$
$$\tilde{R}_0, \tilde{R}_1 \in C[0,T).$$

We then define

$$\tilde{R}(t) = \int_0^1 \left(x - \tfrac{1}{2}\right) \tilde{\phi}(x,t)\, dx + \tilde{R}_0(t) + \tilde{R}_1(t)$$

and consider the approximating problem

$$\phi_t = -(M(x,t,\tilde{R}(t))\phi(x,t))_x + \tfrac{1}{2}(V(x,t,\tilde{R}(t))\phi(x,t))_{xx}$$
$$\phi\big|_{t=0} = \phi_0(x).$$

The smoothness of $\tilde{\phi}$, $\tilde{R}_0$, and $\tilde{R}_1$ imply that there is a constant γ so that

$$|\tilde{R}(t)| \le \gamma \tag{18}$$

on $[0,T)$. It also implies that the functions

$$t \mapsto M(x,t,\tilde{R}(t)) \tag{19}$$
$$t \mapsto V(x,t,\tilde{R}(t)) \tag{20}$$

are continuous.

Throughout this section we will use the notation $M = M(x,t,\tilde{R}(t))$ and $V = V(x,t,\tilde{R}(t))$.

Proposition 12 *Let $T > 0$, and suppose that $\|\phi_0\|_{B_0} < \infty$. Then there exists a unique function $\phi \in C([0,T); B_0) \cap L_2(0,T; B_1)$ so that*

$$\phi_t = -(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx} \qquad \textit{weakly} \tag{21}$$
$$\phi\big|_{t=0} = \phi_0(x). \tag{22}$$

Moreover

$$\sup_{0\le t\le T} \int_0^1 x(1-x)\phi^2\, dx + \int_0^T \int_0^1 [x(1-x)\phi]_x^2\, dx\, dt \le C\, \|\phi_0\|_{B_0}^2 \tag{23}$$

where C depends only on γ and T.

Further, if $\|\phi_0\|_{B_1} < \infty$, then $\phi \in C([0,T); B_1) \cap L_2(0,T; B_2)$ and

$$\sup_{0\le t\le T} \int_0^1 [x(1-x)\phi]_x^2\, dx + \int_0^T \int_0^1 x(1-x)[x(1-x)\phi]_{xx}^2\, dx\, dt \le C\, \|\phi_0\|_{B_1}^2 \tag{24}$$

where again C depends only on γ and T.

Remark: No boundary conditions are being applied to the problem, save through the requirement that $\phi \in L_2(0, T; B_1)$.

Proof (sketch): This follows the usual Galerkin procedure. Indeed, suppose that $\|\phi_0\|_{B_0} < \infty$ and let $\{\phi_k\}$ be the orthonormal basis of B_0 constructed previously. Define $\phi^N(x,t) = \sum_{k=1}^N c_k^N(t)\phi_k(x)$ where the coefficients $c_k^N(t)$ are chosen to satisfy

$$\begin{aligned} \int_0^1 x(1-x)\frac{\partial \phi^N}{\partial t}\phi_j\,dx = &-\int_0^1 (M\phi^N)_x x(1-x)\phi_j\,dx \\ &- \frac{1}{2}\int_0^1 (V\phi^N)_x[x(1-x)\phi_j]_x\,dx \end{aligned} \tag{25}$$

$$c_j^N(0) = \langle \phi_0(\cdot), \phi_j \rangle_{B_0} \tag{26}$$

for each $1 \le j \le N$. Continuity of the coefficients ensures that this system of ordinary differential equations has a solution. Multiplying by c_j^N, summing over j, and strongly using the fact that $m(x,t,\tilde{R}(t))$ and $v(x,t,\tilde{R}(t))$ are smooth in x and can be estimated solely in terms of γ, we obtain

$$\begin{aligned} \frac{d}{dt}\int_0^1 x(1-x)(\phi^N)^2\,dx &+ \int_0^1 [x(1-x)\phi^N]_x^2\,dx \\ &\le C\int_0^1 [x(1-x)]^2(\phi^N)^2\,dx \le C\int_0^1 x(1-x)(\phi^N)^2\,dx \end{aligned}$$

for each t, where C depends only on γ. Gronwall's inequality then gives us (23), at least for ϕ^N.

The existence, uniqueness, and continuity of ϕ with values in B_0 then follows by standard methods; see [15, Chp. III, §4].

If we now assume that $\|\phi_0\|_{B_1} < \infty$, then multiplying (25) by $\lambda_j c_j^N$ and summing, we find that

$$\begin{aligned} \int_0^1 x(1-x)\frac{\partial \phi^N}{\partial t}[x(1-x)\phi^N]_{xx}\,dx = &-\int_0^1 (M\phi^N)_x x(1-x)[x(1-x)\phi^N]_{xx}\,dx \\ &+ \frac{1}{2}\int_0^1 (V\phi^N)_{xx} x(1-x)[x(1-x)\phi^N]_{xx}\,dx. \end{aligned}$$

Integrating by parts in the first term and using the fact that $m(x,t,\tilde{R}(t))$ and $v(x,t,\tilde{R}(t))$ are smooth in x, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\int_0^1 [x(1-x)\phi^N]_x^2\,dx &+ \int_0^1 x(1-x)[x(1-x)\phi^N]_{xx}^2\,dx \\ &\le C\int_0^1 [x(1-x)\phi^N]_x^2\,dx + C\int_0^1 x(1-x)(\phi^N)^2\,dx. \end{aligned}$$

and so Gronwall's inequality gives us (24), at least for ϕ^N. The usual techniques for passage to the limit give us our result. ■

The following is a simple consequence of the existence theorem and the properties of the spaces B_1 and B_2.

Corollary 13 *$x(1-x)\phi \in C([0,T); C^{1/2}[0,1])$, $\phi \in C_{loc}((0,1)\times[0,T))$, and $\phi_t \in L_2(0,T;B_0)$. Further, there is a constant C depending only on γ and T so that*

$$\sup_{0\le t<T} |\phi(x,t)| \le C \max\left(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{x-1}}\right) \|\phi_0\|_{B_1}$$

and for any $1 \le p < 2$ there is a constant C depending only on γ, T and p so that

$$\sup_{0\le t<T} \|\phi(\cdot,t)\|_{L_p(0,1)} \le C \|\phi_0\|_{B_1}.$$

Indeed, this follows from Lemma 5, Corollary 6, and the equation itself.

We need to understand the behavior of the solution near $x=0$ and $x=1$; in particular to interpret (8)-(9) even weakly, we need to be able to estimate $(V\phi)_x\big|_{x=0}$ and $(V\phi)_x\big|_{x=1}$. However the regularity theory developed thus far is insufficient to show that these quantities exist, even in the sense of traces. In particular, we know that there exist functions $f \in B_2$ so that $[x(1-x)f]_x$ is infinite when $x=0$. Fortunately, the equation will provide just enough additional information to allow us to interpret $(V\phi)_x$ when $x=0$ and $x=1$.

To do so, we begin by obtaining precise quantitative bounds on the regularity of the solution in time.

Lemma 14 *Let ϕ be the solution of Proposition 12. Then $\phi \in C^{1/2}([0,T);B_0)$ and there is a constant $C = C(\gamma,T)$ so that*

$$\|\phi(\cdot,t_2)-\phi(\cdot,t_1)\|_{B_0} \le C|t_2-t_1|^{1/2} \|\phi_0\|_{B_1}$$

for any $0 \le t_1 < t_2 < T$.

Proof: Since $\phi \in C([0,T);B_0)$ and $\phi_t \in L_2(0,T;B_0)$, we know that

$$\int_0^1 x(1-x)[\phi(x,t_2)-\phi(x,t_1)]\psi dx = \int_{t_1}^{t_2}\int_0^1 \{-(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx}\}x(1-x)\psi dx dt$$

for any $\psi \in B_0$ and any $0 \le t_1 < t_2 < T$. Setting $\psi(x) = [\phi(x,t_2)-\phi(x,t_1)]$ and using (23) and (24), we find

$$\begin{aligned}\|\phi(\cdot,t_2)-\phi(\cdot,t_1)\|_{B_0}^2 \le & \left(\int_{t_1}^{t_2}\int_0^1 [mx(1-x)\phi]_x^2 \, dx\, dt\right)^{1/2} \\ & \cdot \left(\int_{t_1}^{t_2}\int_0^1 x(1-x)[\phi(x,t_2)-\phi(x,t_1)]^2 \, dx\, dt\right)^{1/2} \\ & + \left(\int_{t_1}^{t_2}\int_0^1 [vx(1-x)\phi]_x^2 \, dx\, dt\right)^{1/2} \\ & \cdot \left(\int_{t_1}^{t_2}\int_0^1 [x(1-x)(\phi(x,t_2)-\phi(x,t_1))]_x^2 \, dx\, dt\right)^{1/2} \\ \le & \, C|t_2-t_1| \|\phi_0\|_{B_1}^2.\end{aligned}$$

■

Lemma 15 *Let ϕ be the solution of Proposition 12. Then $\phi \in C^\alpha([0,T); L_p)$ for any $1 \le p < 2$, and any $0 < \alpha < \frac{1}{p} - \frac{1}{2}$. Further, there is a constant $C = C(\gamma, T, \alpha, p)$ so that*

$$\|\phi(\cdot, t_2) - \phi(\cdot, t_1)\|_{L_p} \le C|t_2 - t_1|^\alpha \|\phi_0\|_{B_1}$$

for all $0 \le t_1 < t_2 < T$.

Proof: Let $0 < \alpha < \frac{1}{p} - \frac{1}{2}$, and let $0 \le t_1 < t_2 < T$.

$$\int_0^{1/2} |\phi(x,t_2) - \phi(x,t_1)|^p \, dx$$
$$\le C \sup_{0 \le x \le 1/2} |x(1-x)\phi^2(x,t_2) + x(1-x)\phi^2(x,t_1)|^{p(1-2\alpha)/2}$$
$$\cdot \int_0^{1/2} |\phi(x,t_2) - \phi(x,t_1)|^{2p\alpha} x^{-p(1-2\alpha)/2} \, dx.$$

Apply Corollary 13 to the first factor and Hölder's inequality to the second to find that

$$\int_0^{1/2} |\phi(x,t_2) - \phi(x,t_1)|^p \, dx$$
$$\le C \|\phi_0\|_{B_1}^{p(1-2\alpha)} \|\phi(\cdot,t_2) - \phi(\cdot,t_1)\|_{B_0}^{2p\alpha} \left(\int_0^{1/2} x^{-\frac{p}{2}\frac{1}{1-p\alpha}} \, dx \right)^{1-p\alpha}.$$

Our conditions on α guarantee that the last integral converges. Repeating the process on the interval $[\frac{1}{2}, 1]$ and using Lemma 14 gives us the result. ■

Define the auxiliary function

$$\nu(x,t) = \int_0^t (V\phi)_x(x,s) \, ds. \tag{27}$$

Then there exists a constant $C = C(\gamma, T)$ so that

$$\sup_{0 \le t < T} \left\{ \|\nu(\cdot,t)\|_{L_2(0,1)} + \left\| \frac{\partial \nu}{\partial t}(\cdot,t) \right\|_{L_2(0,1)} \right\} \le C \|\phi_0\|_{B_1}.$$

Indeed, we know $\phi \in C([0,T); B_1)$ and $t \mapsto V(x,t)$ is continuous, so $\nu_t = (V\phi)_x = (vx(1-x)\phi)_x$ while the estimate follows from (24).

The following Lemma is the key to interpreting $(V\phi)_x$ on the boundary $x = 0$ or $x = 1$. It strongly uses the fact that ϕ solves the equation (21).

Lemma 16 *Let ϕ be the solution of Proposition 12, and let ν be given by* (27). *Then for any $1 \le p < 2$, there exists a constant $C = C(\gamma, T, p)$ so that*

$$\sup_{0 \le t < T} \left\| \frac{\partial \nu}{\partial x}(\cdot,t) \right\|_{L_p(0,1)} \le C \|\phi_0\|_{B_1}.$$

Further, for any $0 < \alpha < \frac{1}{p} - \frac{1}{2}$

$$\frac{\partial \nu}{\partial x} \in C^{\alpha}([0,T); L_p)$$

and there is a constant $C = C(\gamma, T, \alpha, p)$ *so that*

$$\left\| \frac{\partial \nu}{\partial x}(\cdot, t_2) - \frac{\partial \nu}{\partial x}(\cdot, t_1) \right\|_{L_p} \leq C|t_2 - t_1|^{\alpha} \|\phi_0\|_{B_1}.$$

Proof: Because $\phi_t \in L_2(0,T;B_0)$, and $\phi \in C([0,T);B_1)$ we see that

$$\phi(x,t) - \phi_0(x) = \int_0^t \{-(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx}\}\, ds$$

as elements of B_0. Thus

$$\frac{\partial \nu}{\partial x}(x,t) = \int_0^t (V\phi)_{xx}(x,s)\, ds = 2\int_0^t (M\phi)_x(x,s)\, ds + 2[\phi(x,t) - \phi_0(x)].$$

Then

$$\left\| \frac{\partial \nu}{\partial x}(\cdot, t) \right\|_{L_p}^p \leq C \int_0^1 \left(\int_0^t (M\phi)_x(x,s)\, ds \right)^p dx + C\|\phi(\cdot,t)\|_{L_p}^p + C\|\phi_0\|_{L_p}^p.$$

However

$$\int_0^1 \left(\int_0^t (M\phi)_x(x,s)\, ds \right)^p dx \leq \left(\int_0^t \int_0^1 (M\phi)_x^2\, dx\, ds \right)^{p/2} t^{1-p/2}$$

$$\leq t \left(\operatorname*{ess\,sup}_{0 \leq s \leq t} \int_0^1 [mx(1-x)\phi]_x^2\, dx \right)^{p/2} \leq C\|\phi_0\|_{B_1}^p$$

so that $\nu_x \in L_p(0,1)$ for any $1 \leq p < 2$. Further,

$$\left\| \frac{\partial \nu}{\partial x}(\cdot, t_2) - \frac{\partial \nu}{\partial x}(\cdot, t_1) \right\|_{L_p} \leq C \int_0^1 \left(\int_{t_1}^{t_2} (M\phi)_x(x,s)\, ds \right)^p dx$$
$$+ C\|\phi(\cdot,t_2) - \phi(\cdot,t_1)\|_{L_p}^p.$$

Estimating the first term in the same fashion, we see that

$$\left\| \frac{\partial \nu}{\partial x}(\cdot, t_2) - \frac{\partial \nu}{\partial x}(\cdot, t_1) \right\|_{L_p}^p \leq C|t_2 - t_1|\, \|\phi_0\|_{B_1}^p + C|t_2 - t_1|^{\alpha p} \|\phi_0\|_{B_1}^p$$

and since $\alpha \leq \frac{1}{p} - \frac{1}{2} < \frac{1}{p}$, the result follows. ■

As a consequence of this lemma and the Sobolev embedding $W_p^1(0,1) \hookrightarrow C^{1-1/p}[0,1]$, we have the following.

Lemma 17 *Let ϕ be the solution of Proposition 12, and let ν be defined by* (27)*. Then for any $1 \le p < 2$ and any $0 < \alpha < \frac{1}{p} - \frac{1}{2}$*

$$\nu \in C^{\alpha}([0,T); C^{1-\frac{1}{p}}[0,1]).$$

There is a constant C depending only on γ and T so that

$$\sup_{0 \le t < T} \sup_{x \in [0,1]} |\nu(x,t)| \le C \|\phi_0\|_{B_1}$$

and

$$|\nu(x_2,t_2) - \nu(x_1,t_1)| \le C \left\{ |t_2 - t_1|^{\alpha} + |x_2 - x_1|^{1-1/p} \right\} \|\phi_0\|_{B_1}$$

Further, both $\nu(0,t)$ and $\nu(1,t)$ are defined for $0 \le t < T$ and there is a constant $C = C(\gamma, T)$ so that

$$\sup_{0 \le t < T} \left\{ |\nu(0,t)| + |\nu(1,t)| \right\} \le C \|\phi_0\|_{B_1}.$$

Finally, for any $0 < \beta < \frac{1}{2}$, both $\nu(0,t) \in C^{\beta}[0,T)$ and $\nu(1,t) \in C^{\beta}[0,T)$ and there is a constant $C = C(\gamma, T, \beta)$ so that

$$\left|\nu(0,t_2) - \nu(0,t_1)\right| \le C|t_2 - t_1|^{\beta} \|\phi_0\|_{B_1}$$
$$\left|\nu(1,t_2) - \nu(1,t_1)\right| \le C|t_2 - t_1|^{\beta} \|\phi_0\|_{B_1}$$

for any $0 \le t_1 < t_2 < T$.

Up to this point, all of our estimates have depended on γ from (18). Now we turn to estimates that are independent of γ. We begin with the following weak maximum principle.

Proposition 18 *Let ϕ be the solution of Proposition 12. Then for any $0 \le t_1 < t_2 < T$*

$$\int_0^1 \phi^{\pm}(x,t_2)\, dx \le \int_0^1 \phi^{\pm}(x,t_1)\, dx.$$

Proof: Let $\epsilon > 0$, let $0 < a < b < 1$, and assume that $t_1 > 0$. Consider

$$\psi = \pm \frac{\phi^{\pm}}{x(1-x)\phi^{\pm} + \epsilon}.$$

Clearly $\psi \in L_2(0,T;B_0)$ as $\|\psi\|^2_{L_2(0,T;B_0)} \le \epsilon^{-2} \|\phi\|^2_{L_2(0,T;B_0)}$. Let ζ_n be smooth functions with $\zeta_n \to \chi_{[a,b]\times[t_1,t_2]}$. Take the inner product of the equation with $\psi\zeta_n$ in B_0, integrate in time, and send $n \to \infty$ to see that

$$\begin{aligned}
\int_{t_1}^{t_2} \int_a^b \phi_t^{\pm} \frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm} + \epsilon}\, dx\, dt \\
= - \int_{t_1}^{t_2} \int_a^b (M\phi^{\pm})_x \frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm} + \epsilon}\, dx\, dt \\
\pm \frac{1}{2} \int_{t_1}^{t_2} \int_a^b (V\phi)_{xx} \frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm} + \epsilon}\, dx\, dt. \quad (28)
\end{aligned}$$

Because

$$\phi_t^{\pm}\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}=\frac{\partial}{\partial t}\left\{\phi^{\pm}-\epsilon\ln\big(x(1-x)\phi^{\pm}+\epsilon\big)\right\},$$

we have

$$\int_{t_1}^{t_2}\int_a^b \phi_t^{\pm}\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}\,dx\,dt \\ =\int_a^b\left\{\phi^{\pm}-\frac{\epsilon}{x(1-x)}\ln\big(x(1-x)\phi^{\pm}+\epsilon\big)\right\}\,dx\Bigg|_{t_1}^{t_2},$$

where we have used the fact that the last integrand is an element of $C^{\beta}([0,T);L_1)$ for any $0<\beta<1/2$ (Lemma 15). Send $\epsilon\downarrow 0$ to find

$$\lim_{\epsilon\downarrow 0}\int_{t_1}^{t_2}\int_a^b \phi_t^{\pm}\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}\,dx\,dt=\int_a^b\phi^{\pm}(x,t)\,dx\Bigg|_{t_1}^{t_2}. \tag{29}$$

The middle term in (28) is estimated simply; indeed, because

$$\left\|(M\phi^{\pm})_x\right\|_{L_1(0,T;L_1(0,1))} \\ \le\int_0^T\int_0^1\left|m[x(1-x)\phi^{\pm}]_x+m_x x(1-x)\phi^{\pm}\right|\,dx\,dt \\ \le C\left\|\phi\right\|_{L_2(0,T;B_1)}$$

and because

$$0\le\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}\le 1$$

we can use dominated convergence to find

$$\lim_{\epsilon\downarrow 0}\int_{t_1}^{t_2}\int_a^b(M\phi^{\pm})_x\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}\,dx\,dt=\int_{t_1}^{t_2}\int_a^b(M\phi^{\pm})_x\,dx\,dt. \tag{30}$$

To estimate the last term, we note that V is smooth in x while $\phi\in L_2(0,T;B_2)\hookrightarrow L_2(0,T;C^{3/2}_{\text{loc}}(0,1))$ so that we can integrate by parts and find

$$\pm\int_{t_1}^{t_2}\int_a^b(V\phi)_{xx}\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}\,dx\,dt \\ =\pm\int_{t_1}^{t_2}(V\phi)_x\frac{x(1-x)\phi^{\pm}}{x(1-x)\phi^{\pm}+\epsilon}\,dt\Bigg|_{x=a}^{x=b} \\ -\int_{t_1}^{t_2}\int_a^b(V\phi^{\pm})_x\frac{\epsilon[x(1-x)\phi^{\pm}]_x}{(x(1-x)\phi^{\pm}+\epsilon)^2}\,dx\,dt.$$

There exists a constant C depending only on γ so that

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_a^b (V\phi^\pm)_x \frac{\epsilon[x(1-x)\phi^\pm]_x}{(x(1-x)\phi^\pm+\epsilon)^2}\,dx\,dt \\
&\quad= \int_{t_1}^{t_2}\int_a^b v\frac{\epsilon[x(1-x)\phi^\pm]_x^2}{(x(1-x)\phi^\pm+\epsilon)^2}\,dx\,dt \\
&\qquad+ \int_{t_1}^{t_2}\int_a^b v_x\frac{\epsilon x(1-x)\phi^\pm[x(1-x)\phi^\pm]_x}{(x(1-x)\phi^\pm+\epsilon)^2}\,dx\,dt \\
&\quad\geq -C\epsilon\int_{t_1}^{t_2}\int_a^b \frac{|[x(1-x)\phi^\pm]_x|}{x(1-x)\phi^\pm+\epsilon}\,dx\,dt
\end{aligned}$$

so that

$$\begin{aligned}
\pm&\int_{t_1}^{t_2}\int_a^b (V\phi_n)_{xx}\frac{\phi_n^\pm}{x(1-x)\phi_n^\pm+\epsilon}\,dx\,dt \\
&\leq \pm\int_{t_1}^{t_2}(V\phi_n)_x\frac{\phi_n^\pm}{x(1-x)\phi_n^\pm+\epsilon}\,dt\Bigg|_{x=a}^{x=b} \\
&\qquad + C\epsilon\int_{t_1}^{t_2}\int_a^b\frac{|[x(1-x)\phi_n^\pm]_x|}{x(1-x)\phi_n^\pm+\epsilon}\,dx\,dt. \qquad (31)
\end{aligned}$$

Now we wish to pass to the limit as $\epsilon\downarrow 0$. To handle the first term, we note that $\phi\in C([0,T);B_1)$, so $V\phi\in C([0,T);W_2^1)$, and thus $V\phi^\pm\in C([0,T);W_2^1)$. As a consequence $(V\phi^\pm)_x=\pm(V\phi)_x\chi[\phi^\pm>0]$ is well-defined as an element of $C([0,T);L_2)$. However, this, by itself, is insufficient to define $(V\phi^\pm)_x$ for any particular x, as we need. On the other hand, because $\phi\in L_2(0,T;B_2)\hookrightarrow L_2(0,T;C^{3/2}_{loc}(0,1))$, the function $\pm(V\phi)_x\chi[\phi^\pm>0]$ is defined for all x as an element of $L_2(0,T)$. This lets us use dominated convergence in the first term, and find

$$\lim_{\epsilon\downarrow 0}\int_{t_1}^{t_2}\pm(V\phi)_x\frac{x(1-x)\phi^\pm}{x(1-x)\phi^\pm+\epsilon}\,dt\Bigg|_{x=a}^{x=b}=\int_{t_1}^{t_2}\pm(V\phi)_x\chi[\phi^\pm>0]\,dt\Bigg|_{x=a}^{x=b}.$$

For the second, we note that, because $0\leq\frac{\epsilon}{x(1-x)\phi^\pm+\epsilon}\leq 1$ and because $\phi\in L_2(0,T;B_2)$, we can apply dominated convergence to see that

$$\lim_{\epsilon\downarrow 0}\int_{t_1}^{t_2}\int_a^b\frac{\epsilon}{x(1-x)\phi^\pm+\epsilon}|[x(1-x)\phi^\pm]_x|\,dx\,dt=0.$$

Thus

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}&\int_{t_1}^{t_2}\int_a^b\pm(V\phi)_{xx}\frac{\phi^\pm}{x(1-x)\phi^\pm+\epsilon}\,dx\,dt \\
&\leq\int_{t_1}^{t_2}\pm(V\phi)_x\chi[\phi^\pm>0]\,dt\Bigg|_{x=a}^{x=b}. \qquad (32)
\end{aligned}$$

Combining (28), (29), (30) and (32) we find that, for any $0 < a < b < 1$ and any $0 < t_1 < t_2 < T$

$$\int_a^b \phi^\pm \, dx \Big|_{t=t_1}^{t=t_2} \leq \int_{t_1}^{t_1} \int_a^b (M\phi^\pm)_x \, dx \, dt \pm \int_{t_1}^{t_2} (V\phi)_x \chi[\phi^\pm > 0] \, dt \Big|_{x=a}^{x=b}. \quad (33)$$

Next, we would like to send $a \downarrow 0$ and $b \uparrow 1$. Because $\phi \in C^\beta([0,T); L_1)$ for any $0 < \beta < 1/2$, dominated convergence implies

$$\lim_{b\uparrow 1} \lim_{a \downarrow 0} \int_a^b \phi^\pm(x,t) \, dx = \int_0^1 \phi(x,t)^\pm \, dx \quad (34)$$

for any $0 \leq t < T$.

For the second term of (33) we start with the fact that $\phi \in C([0,T); B_1)$, so the continuity of $M(x,t,\tilde{R}(t))$ in time implies $M\phi \in C([0,T); W_2^1(0,1))$ and hence $M\phi^\pm \in C([0,T); W_2^1(0,1))$. Thus

$$\lim_{b\uparrow 1} \lim_{a \downarrow 0} \int_{t_1}^{t_2} \int_a^b (M\phi^\pm)_x \, dx \, dt = \int_{t_1}^{t_2} \int_0^1 (M\phi^\pm)_x \, dx \, dt.$$

For each fixed t, we know $M\phi = mx(1-x)\phi$, so $(M\phi)(\cdot, t) \in \overset{\circ}{W}{}_2^1(0,1)$ and thus $(M\phi^\pm)(\cdot,t) \in \overset{\circ}{W}{}_2^1(0,1)$. Consequently for each t

$$\int_0^1 (M\phi^\pm)_x \, dx = 0. \quad (35)$$

Though the embedding $\phi \in L_2(0,T;B_2) \hookrightarrow L_2(0,T; C^{3/2}_{\text{loc}}(0,1))$ is sufficient to define $\pm(V\phi)_x \chi[\phi^\pm > 0]$ for all x, it is not sufficient to ensure its continuity in x. Despite this, we can still find sequences $a_n \downarrow 0$ and $b_n \uparrow 1$ on which we can bound $\lim_{n\to\infty} \int_{t_1}^{t_2} \pm(V\phi)_x \chi[\phi^\pm > 0] \, dt \big|_{x=a_n}^{x=b_n}$. Indeed for fixed t_1 and t_2, define

$$\mu^\pm(x) = \int_{t_1}^{t_2} (V\phi^\pm)(x,t) \, dt.$$

Because $V \geq 0$, we see that $\mu^\pm(x) \geq 0$ for all x. Now $\mu^\pm \in W_2^1(0,1)$; indeed $\|\mu^\pm\|_{W_2^1(0,1)} \leq C \|\phi\|_{L_2(0,T;B_1)}$. Thus

$$\mu_x^\pm(x) = \int_{t_1}^{t_2} \pm(V\phi)_x \chi[\phi^\pm > 0] \, dt$$

for almost every x. Further, Sobolev embedding implies $\mu^\pm \in C^{1/2}[0,1]$. and $\mu^\pm(0) = \mu^\pm(1) = 0$; indeed Corollary 6 implies

$$\mu^\pm(x) = \int_{t_1}^{t_2} vx(1-x)\phi^\pm \, dt$$
$$\leq Cx(1-x) \max\left(\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{1-x}}\right) \|\phi\|_{L_2(0,T;B_1)}.$$

Now we claim that for every $\delta > 0$, both

$$\text{meas}\{x \in (0,\delta) : \mu_x^{\pm}(x) \geq 0\} > 0 \text{ and} \tag{36a}$$
$$\text{meas}\{x \in (1-\delta,1) : \mu_x^{\pm}(x) \leq 0\} > 0. \tag{36b}$$

Indeed, if the first does not hold, then there exists some $\delta > 0$ so that $\mu_x^{\pm} < 0$ for almost every $x \in (0,\delta)$. Then

$$\mu^{\pm}(x) = \int_0^x \mu_x^{\pm}(y)\, dy < 0$$

on $(0,\delta)$, which contradicts the fact that $\mu^{\pm}(x) \geq 0$. The second claim follows similarly.

As a consequence, we can find sequences $a_n \downarrow 0$ and $b_n \uparrow 1$ so that

$$\pm \int_{t_1}^{t_2} (V\phi)_x \chi[\phi^{\pm} > 0]\, dt\Big|_{x=a_n} \geq 0 \tag{37a}$$
$$\pm \int_{t_1}^{t_2} (V\phi)_x \chi[\phi^{\pm} > 0]\, dt\Big|_{x=b_n} \leq 0. \tag{37b}$$

If we then pass to the limit in (33) along the sequences $a_n \downarrow 0$ and $b_n \uparrow 1$ and apply (34), (35) and (37), we see that

$$\int_0^1 \phi^{\pm}(x,t)\, dx\Big|_{t=t_1}^{t=t_2} \leq 0$$

at least if $t_1 > 0$. The result for $t_1 = 0$ follows from this and the continuity $\phi \in C^{\beta}([0,T];L_1)$ for $0 < \beta < 1/2$. ■

Based on (8)-(9), we make the definition

$$R_0(t) = R_0(0) - \frac{1}{4}\nu(0,t) \qquad \text{and} \qquad R_1(t) = R_1(0) - \frac{1}{4}\nu(1,t). \tag{38}$$

We can then define

$$R(t) = \int_0^1 (x - \tfrac{1}{2})\phi(x,t)\, dx + R_0(t) + R_1(t). \tag{39}$$

Lemma 19 *Let ϕ be the solution of Proposition 12. Then for any $0 \leq t_1 < t_2 < T$*

$$R(t_2) - R(t_1) = \int_{t_1}^{t_2}\int_0^1 M\phi\, dx\, dt.$$

Proof: Our equation is satisfied in $L_2(0,T;B_0) \hookrightarrow L_2(0,T;L_{2,\text{loc}}(0,1))$, so for any $0 < a < b < 1$

$$\int_{t_1}^{t_2}\int_a^b (x - \tfrac{1}{2})\phi_t\, dx\, dt = -\int_{t_1}^{t_2}\int_a^b (x - \tfrac{1}{2})(M\phi)_x\, dx\, dt$$
$$+ \frac{1}{2}\int_{t_1}^{t_2}\int_a^b (x - \tfrac{1}{2})(V\phi)_{xx}\, dx\, dt.$$

Now $\phi \in C^\beta([0,T); L_1(0,1))$ for any $0 < \beta < 1/2$, so that

$$\lim_{b\uparrow 1}\lim_{a\downarrow 0}\int_{t_1}^{t_2}\int_a^b (x-\tfrac{1}{2})\phi_t \, dx\, dt = \int_0^1 (x-\tfrac{1}{2})\phi(x,t)\, dx\Big|_{t=t_1}^{t=t_2}. \tag{40}$$

For the second term, because $\phi \in C([0,T); B_1)$ we know $M\phi \in C([0,T); \overset{\circ}{W}{}_2^1(0,1))$. We can then pass to the limit and integrate by parts to find

$$-\lim_{b\uparrow 1}\lim_{a\downarrow 0}\int_{t_1}^{t_2}\int_a^b (x-\tfrac{1}{2})(M\phi)_x \, dx\, dt$$
$$= \int_{t_1}^{t_2}\int_0^1 (x-\tfrac{1}{2})(M\phi)_x \, dx\, dt = \int_{t_1}^{t_2}\int_0^1 M\phi \, dx\, dt. \tag{41}$$

For the last term, we start by noting that

$$\int_{t_1}^{t_2}\int_a^b (x-\tfrac{1}{2})(V\phi)_{xx}\, dx\, dt = \int_a^b (x-\tfrac{1}{2})[\nu_x(x,t_2)-\nu_x(x,t_1)]\, dx.$$

Now for any $1 \le p < 2$ and any $0 < \alpha < 1/p - 1/2$ we have $\nu_x \in C^\alpha([0,T); L_p(0,1))$ and $\nu \in C^\alpha([0,T); C^{1-1/p}[0,1])$, so an integration by parts gives us

$$\lim_{b\uparrow 1}\lim_{a\downarrow 0}\int_{t_1}^{t_2}\int_a^b (x-\tfrac{1}{2})(V\phi)_{xx}\, dx\, dt = -\int_0^1 [\nu(x,t_2)-\nu(x,t_1)]\, dx$$
$$+ \tfrac{1}{2}[\nu(1,t_2)-\nu(1,t_1)] + \tfrac{1}{2}[\nu(0,t_2)-\nu(0,t_1)].$$

Now $\phi \in C([0,T); B_1)$, so $V\phi \in C([0,T); \overset{\circ}{W}{}_2^1(0,1))$ and hence

$$\int_0^1 [\nu(x,t_2)-\nu(x,t_1)]\, dx = \int_{t_1}^{t_2}\int_0^1 (V\phi)_x \, dx\, dt = 0.$$

Thus

$$\lim_{b\uparrow 1}\lim_{a\downarrow 0}\int_{t_1}^{t_2}\int_a^b (x-\tfrac{1}{2})(V\phi)_{xx}\, dx\, dt$$
$$= \tfrac{1}{2}[\nu(1,t_2)-\nu(1,t_1)] + \tfrac{1}{2}[\nu(0,t_2)-\nu(0,t_1)]$$
$$= -2(R_1(t_2)-R_1(t_1)) - 2(R_0(t_2)-R_0(t_1)). \tag{42}$$

Combining (40), (41), and (42) gives us our result. ■

Lemma 20 *Let ϕ be the solution of Proposition 12, and suppose that $\phi_0 \ge 0$. Then $R \in C^1[0,T)$ and*

$$|R(t)| \le \left[|R(0)| + \|\phi_0\|_{L_1(0,1)}\int_0^t \mathcal{M}_1(s)\, ds\right]\exp\left[\|\phi_0\|_{L_1(0,1)}\int_0^t \mathcal{M}_2(s)\, ds\right]$$

for any $0 \le t < T$.

Proof: Because $\phi_0 \geq 0$, we can use Proposition 18 to conclude that $\phi \geq 0$. Then Lemma 19 implies

$$R(t) = R(0) + \int_0^t \int_0^1 M\phi \, dx \, ds.$$

Applying (H3), we see that

$$|R(t)| \leq |R(0)| + \int_0^t [\mathcal{M}_1(s) + \mathcal{M}_2(s)|R(s)|] \int_0^1 \phi(x,s) \, dx \, ds.$$

Then because Proposition 18 implies $\|\phi(\cdot,t)\|_{L_1(0,1)} \leq \|\phi_0\|_{L_1(0,1)}$, we find

$$|R(t)| \leq \left[|R(0)| + \|\phi_0\|_{L_1(0,1)} \int_0^t \mathcal{M}_1(s) \, ds\right] + \int_0^t \|\phi_0\|_{L_1(0,1)} \mathcal{M}_2(s)|R(s)| \, ds$$

and so Gronwall's inequality [3, Thm. 2.1] gives us the result. ■

6 The fixed point argument

To prove that the full nonlinear problem has a solution, we will rely on the compactness properties of the set of solutions to the problem with fixed coefficients. In particular, we will need the following.

Lemma 21 *Let* $0 < \alpha < \frac{1}{2}$, *let* $\{\phi_n\}_{n=1}^\infty \subset C([0,T);B_1) \cap C^\alpha([0,T);L_1)$, *and suppose that there is a constant* C *so that, for all* n

$$\sup_{0 \leq t < T} \|\phi_n(\cdot,t)\|_{B_1} \leq C \tag{43}$$

and for every $0 \leq t_1 < t_2 < T$ *and for all* n

$$\|\phi_n(\cdot,t_2) - \phi_n(\cdot,t_1)\|_{L_1} \leq C|t_2 - t_1|^\alpha. \tag{44}$$

Then there is a subsequence $\{\phi_{n_j}\}_{j=1}^\infty$ *and a function* $\phi \in C^\alpha([0,T);L_1)$ *so that*

$$\left\|\phi_{n_j}(\cdot,t) - \phi(\cdot,t)\right\|_{L_1} \longrightarrow 0$$

uniformly for $t \in [0,T)$.

Proof: Let $\mathcal{T} = \{t_j\}_{j=1}^\infty$ be a dense subset of $[0,T)$. We claim that, for each t_j, there is a function $\phi(\cdot,t_j) \in B_1 \hookrightarrow L_1(0,1)$ and a sequence $\{n_j(k)\}_{k=1}^\infty$ so that

- $\{n_{j+1}(k)\}_{k=1}^\infty$ is a subsequence of $\{n_j(k)\}_{k=1}^\infty$ for all $j = 1, 2, \ldots$; and
- $\left\|\phi_{n_j(k)}(\cdot,t_j) - \phi(\cdot,t_j)\right\|_{L_1} \leq 2^{-k}$ for all j and k.

This follows by induction. Indeed, (43) implies that $\|\phi_n(\cdot,t_1)\|_{B_1} \leq C$ for all n. Then the compactness $B_1 \hookrightarrow L_1$ implies that there is a function $\phi(\cdot,t_1) \in B_1 \hookrightarrow L_1(0,1)$

and a subsequence $\{n_1(k)\}_{k=1}^\infty$ so that $\left\|\phi_{n_1(k)}(\cdot,t_1)-\phi(\cdot,t_1)\right\|_{L_1} \to 0$; choosing another subsequence if necessary, we can ensure

$$\left\|\phi_{n_1(k)}(\cdot,t_1)-\phi(\cdot,t_1)\right\|_{L_1} < 2^{-k}$$

for all k.

Given $\phi(\cdot,t_j)$ and $\{n_j(k)\}_{k=1}^\infty$, we again use (43) to see that $\left\|\phi_{n_j(k)}(\cdot,t_{j+1})\right\|_{B_1} \leq C$ for all k. Again the compactness $B_1 \hookrightarrow L_1$ allows us to find a function $\phi(\cdot,t_{j+1})$ and a subsequence $\{n_{j+1}(k)\}_{k=1}^\infty$ of $\{n_j(k)\}_{k=1}^\infty$ so that

$$\left\|\phi_{n_{j+1}(k)}(\cdot,t_{j+1})-\phi(\cdot,t_{j+1})\right\|_{L_1} < 2^{-k}$$

for all k. This completes the induction.

Define $n_j = n_j(j)$. Then for any k and for any $j \geq k$

$$\left\|\phi_{n_j}(\cdot,t_k)-\phi(\cdot,t_k)\right\|_{L_1} \leq 2^{-j}. \tag{45}$$

Indeed, because $j \geq k$, we know that $\{n_j(i)\}_{i=1}^\infty$ is a subsequence of $\{n_k(i)\}_{i=1}^\infty$. Thus there is some $i \geq j$ so that

$$n_j(j) = n_k(i).$$

Then

$$\left\|\phi_{n_j}(\cdot,t_k)-\phi(\cdot,t_k)\right\|_{L_1} = \left\|\phi_{n_k(i)}(\cdot,t_k)-\phi(\cdot,t_k)\right\|_{L_1} \leq 2^{-i} \leq 2^{-j},$$

proving the claim.

Let $s,t \in \mathcal{T}$. Then, for any i

$$\begin{aligned}\|\phi(\cdot,t)-\phi(\cdot,s)\|_{L_1} &\leq \|\phi(\cdot,t)-\phi_{n_i}(\cdot,t)\|_{L_1} \\ &\quad + \|\phi_{n_i}(\cdot,t)-\phi_{n_i}(\cdot,s)\|_{L_1} + \|\phi_{n_i}(\cdot,s)-\phi(\cdot,s)\|_{L_1}.\end{aligned}$$

We can use (44) to estimate the middle term, then use (45) and the fact that s and t are fixed to pass to the limit in the first and last to find that

$$\|\phi(\cdot,t)-\phi(\cdot,s)\|_{L_1} \leq C|t-s|^\alpha \tag{46}$$

for all $s,t \in \mathcal{T}$.

Now choose $t \notin \mathcal{T}$. Let $\{\tau_k\}_{k=1}^\infty \subset \mathcal{T}$ with $\tau_k \to t$. Then (46) implies that $\{\phi(\cdot,\tau_k)\}_{k=1}^\infty$ is Cauchy in $L_1(0,1)$. Thus there exists a function $\phi(\cdot,t) \in L_1(0,1)$ so that $\phi(\cdot,\tau_j) \to \phi(\cdot,t)$ in $L_1(0,1)$. Then, passing to the limit in (46), we see that we have defined a function $\phi(\cdot,t)$ for all t so that

$$\|\phi(\cdot,t)-\phi(\cdot,s)\|_{L_1} \leq C|t-s|^\alpha \tag{47}$$

for all s and t.

Now we claim that $\phi_{n_j}(\cdot,t) \to \phi(\cdot,t)$ in $L_1(0,1)$, uniformly in $t \in [0,T)$. Indeed, let $\epsilon > 0$. Let $\{s_i\}_{i=1}^n$ be a finite set of elements of $[0,T)$ so that for all $t \in [0,T)$ there exists i so that $|t-s_i| < \frac{1}{2}\left(\frac{\epsilon}{4C}\right)^{1/\alpha}$. Because $\mathcal{T}$ is dense in $[0,T)$, for each i there exists $k(i)$ so that $|t_{k(i)} - s_i| < \frac{1}{2}\left(\frac{\epsilon}{4C}\right)^{1/\alpha}$. Set $K = \max\{k(i)\}_{i=1}^{n}$. Then K depends only on ϵ and C, and for any $t \in [0,T)$ there exists $t_k \in \mathcal{T}$ so that $|t - t_k| < \left(\frac{\epsilon}{4C}\right)^{1/\alpha}$ with $k \leq K$.

Choose $j \geq K$ so large that $2^{-j} \leq \epsilon/2$. Let $t \in [0,T)$, and choose t_k as above. Then

$$\left\|\phi(\cdot,t) - \phi_{n_j}(\cdot,t)\right\|_{L_1} \leq \|\phi(\cdot,t) - \phi(\cdot,t_k)\|_{L_1} + \left\|\phi(\cdot,t_k) - \phi_{n_j}(\cdot,t_k)\right\|_{L_1} + \left\|\phi_{n_j}(\cdot,t_k) - \phi_{n_j}(\cdot,t)\right\|_{L_1}.$$

Then (44) and (47) allow us to estimate the first and the last terms, while we can use (45) on the last to find

$$\left\|\phi(\cdot,t) - \phi_{n_j}(\cdot,t)\right\|_{L_1} \leq 2C|t - t_k|^{\alpha} + 2^{-j}.$$

Our choices of t_k and j imply the result. ■

Lemma 22 *Let $\{\phi_n\}_{n=1}^{\infty} \subset C([0,T);B_1) \cap C^{1/2}([0,T);B_0)$, and suppose that there is a constant C so that, for all n*

$$\sup_{0\leq t<T} \|\phi_n(\cdot,t)\|_{B_1} \leq C$$

and for every $0 \leq t_1 < t_2 < T$ and for all n

$$\|\phi_n(\cdot,t_2) - \phi_n(\cdot,t_1)\|_{L_1} \leq C|t_2 - t_1|^{1/2}.$$

Then there is a subsequence $\{\phi_{n_j}\}_{j=1}^{\infty}$ and a function $\phi \in C^{1/2}([0,T);B_0)$ so that

$$\left\|\phi_{n_j}(\cdot,t) - \phi(\cdot,t)\right\|_{B_0} \longrightarrow 0$$

uniformly for $t \in [0,T)$.

Proof: This follows the same lines as the previous, with B_0 in place of L_1. ■

For notational convenience, let $\mathcal{U}$ be the space

$$\mathcal{U} = C([0,T);L_1(0,1)) \times C[0,T) \times C[0,T).$$

Consider the function

$$\mathfrak{F} : \mathcal{U} \to \mathcal{U}$$

defined by the rule

$$\mathfrak{F}(\tilde{\phi}, \tilde{R}_0, \tilde{R}_1) = (\phi, R_0, R_1)$$

where ϕ is the solution given by Proposition 12 and R_0 and R_1 are defined by (38). Existence will follow once we show that $\mathfrak{F}$ has a fixed point.

To show that $\mathfrak{F}$ is continuous, let $\{(\tilde{\phi}_k, \tilde{R}_{k,0}, \tilde{R}_{k,1})\}_{k=1}^{\infty} \subset \mathcal{U}$ and suppose that $(\tilde{\phi}_k, \tilde{R}_{k,0}, \tilde{R}_{k,1}) \longrightarrow (\tilde{\phi}, \tilde{R}_0, \tilde{R}_1)$ in $\mathcal{U}$. Let $\mathfrak{F}(\tilde{\phi}_k, \tilde{R}_{k,0}, \tilde{R}_{k,1}) = (\phi_k, R_{k,0}, R_{k,1})$ and $\mathfrak{F}(\tilde{\phi}, \tilde{R}_0, \tilde{R}_1) = (\phi, R_0, R_1)$.

Let

$$\tilde{R}_k(t) = \int_0^1 (x - \tfrac{1}{2})\tilde{\phi}_k(x,t)\,dx + \tilde{R}_{k,0}(t) + \tilde{R}_{k,1}(t).$$

Then clearly $\tilde{R}_k(t) \to \tilde{R}(t)$ where

$$\tilde{R}(t) = \int_0^1 (x - \tfrac{1}{2})\tilde{\phi}(x,t)\,dx + \tilde{R}_0(t) + \tilde{R}_1(t).$$

Moreover, the convergence $(\tilde{\phi}_k, \tilde{R}_{k,0}, \tilde{R}_{k,1}) \longrightarrow (\tilde{\phi}, \tilde{R}_0, \tilde{R}_1)$ in $\mathcal{U}$ implies that there exists a single constant γ_0 so that $|\tilde{R}_k|, |R_k| \le \gamma_0$ for all t.

Define

$$M_k = M(x,t,\tilde{R}_k(t)) \quad \text{and} \quad M = M(x,t,\tilde{R}(t));$$
$$V_k = V(x,t,\tilde{R}_k(t)) \quad \text{and} \quad V = V(x,t,\tilde{R}(t)).$$

Now

$$M_k - M = \left[\frac{\partial M}{\partial R}(x,t,\lambda\tilde{R}_k(t) + (1-\lambda)\tilde{R}(t))\right](\tilde{R}_k(t) - \tilde{R}(t))$$

for some $0 \le \lambda \le 1$. Thus (H2) implies that there is a constant $C = C(\gamma_0)$ so that

$$|M_k - M| \le C|\tilde{R}_k(t) - \tilde{R}(t)|.$$

Similarly,

$$|V_k - V| \le C|\tilde{R}_k(t) - \tilde{R}(t)|.$$

As a consequence, we know $M_k \to M$ and $V_k \to V$ uniformly on $[0,1] \times [0,T)$.

Let $\{k(n)\}_{n=1}^{\infty}$ be any subsequence of $\mathbf{N}$. Then $\left\|\phi_{k(n)}\right\|_{L_2(0,T;B_2)} \le C\left\|\phi_0\right\|_{B_1}$ where C depends only on γ_0. Similarly, for any $0 \le t < T$ we have $\left\|\phi_{k(n)}(\cdot,t)\right\|_{B_1} \le C\left\|\phi_0\right\|_{B_1}$ and for any $0 \le t_1 < t_2 < T$ we have $\left\|\phi_{k(n)}(\cdot,t_2) - \phi_{k(n)}(\cdot,t_1)\right\|_{B_0} \le C|t_2 - t_1|^{1/2}\left\|\phi_0\right\|_{B_1}$ and $\left\|\phi_{k(n)}(\cdot,t_2) - \phi_{k(n)}(\cdot,t_1)\right\|_{L_1} \le C|t_2 - t_1|^{\alpha}\left\|\phi_0\right\|_{B_1}$ for any $0 < \alpha < 1/2$. Thus Lemmas 21 and 22 let us find a function $\phi^* \in L_2(0,T;B_2) \cap C^{1/2}([0,T);B_0) \cap C^{\alpha}([0,T);L_1)$ and a subsequence $\{k'(n)\}_{n=1}^{\infty}$ so that $\phi_{k'(n)} \rightharpoonup \phi^*$ weakly in $L_2(0,T;B_2)$ and $\phi_{k'(n)}(\cdot,t) \to \phi^*(\cdot,t)$ strongly in B_0 and in $L_1(0,1)$ uniformly in t.

Lemma 17 and the Ascoli-Arzela theorem imply that there is a function $\nu^* \in C^{\alpha}([0,T);C^{1-1/p}[0,1])$ so that if

$$\nu_k(x,t) = \int_0^t (V_k\phi_k)_x(x,s)\,dx\,ds$$

then $\nu_{k'(n)} \longrightarrow \nu^*$ uniformly on $[0,1] \times [0,T)$ modulo an additional subsequence which we still call $\{k'(n)\}_{n=1}^{\infty}$.

Because $(\phi_{k'(n)})_t = -(M_{k'(n)}\phi_{k'(n)})_x + \frac{1}{2}(V_{k'(n)}\phi_{k'(n)})_{xx}$, we can pass to the limit as $n \to \infty$ to find that ϕ^* satisfies $\phi^*_t = -(M\phi^*)_x + \frac{1}{2}(V\phi^*)_{xx}$. Further, $\phi^*(\cdot,t) \to \phi_0$ as $t \downarrow 0$ in B_0; this follows from the uniform convergence $\phi_{k'(n)} \to \phi^*$

in $C^{1/2}([0,T];B_0)$ and the fact that $\phi_{k'(n)}(\cdot,t) \to \phi_0$ as $t \downarrow 0$ in B_0 for any fixed $k'(n)$. The uniqueness of Proposition 12 implies $\phi^* = \phi$. Further, $\nu_{k'(n)}(x,t) = \int_0^t (V_{k'(n)}\phi_{k'(n)})_x(x,s)\,ds$, so passing to the limit we find that $\nu^*(x,t) = \nu(x,t)$.

Therefore we have shown that, for any subsequence $\{k(n)\}_{n=1}^{\infty}$, there exists a subsubsequence $\{k'(n)\}_{n=1}^{\infty}$ so that $\phi_{k'(n)} \longrightarrow \phi \quad \text{in } C^{\alpha}([0,T];L_1)$ and $\nu_{k'(n)} \longrightarrow \nu \quad \text{in } C^{\beta}([0,T];C^{1-1/p}[0,1])$ for any $0 < \alpha < \frac{1}{2}$, for any $1 \le p < 2$, and for any $0 < \beta < \frac{1}{2} - \frac{1}{p}$. Thus, we know the original sequence converges, and

$$\begin{aligned} \phi_k &\longrightarrow \phi \qquad \text{in } C^{\alpha}([0,T];L_1) \\ \nu_k &\longrightarrow \nu \qquad \text{in } C^{\beta}([0,T];C^{1-1/p}[0,1]) \end{aligned}$$

Now

$$\begin{aligned} R_{k,0}(t) &= R_0(0) + \nu_k(0,t) \qquad & R_0(t) &= R_0(0) + \nu(0,t) \\ R_{k,1}(t) &= R_1(0) + \nu_k(1,t) \qquad & R_1(t) &= R_1(0) + \nu(1,t) \end{aligned}$$

so that $(\phi_k, R_{k,0}, R_{k,1}) \longrightarrow (\phi, R_0, R_1)$ in $\mathcal{U}$; hence $\mathfrak{F} : \mathcal{U} \to \mathcal{U}$ is continuous.

Compactness of $\mathfrak{F}$ is almost immediate. Indeed, if $\{\tilde{\phi}_k, \tilde{R}_{k,0}, \tilde{R}_{k,1}\} \subset \mathcal{U}$ is bounded then Lemma 17 and Lemma 21 imply that the solutions $\{\phi_k, R_{k,0}, R_{k,1}\}$ have a subsequence that converges in $\mathcal{U}$.

Finally, the set $\{(\phi, R_0, R_1) \in \mathcal{U} : (\phi, R_0, R_1) = \sigma\mathfrak{F}(\phi, R_0, R_1)$ for some $0 \le \sigma \le 1\}$ is bounded in $\mathcal{U}$. Indeed, suppose that $(\phi, R_0, R_1) = \sigma\mathfrak{F}(\phi, R_0, R_1)$, and define $(\phi^*, R_0^*, R_1^*) = \mathfrak{F}(\phi, R_0, R_1)$. Then

$$\begin{cases} \phi^*(x,t) = -(M\phi^*)_x + \frac{1}{2}(V\phi^*)_{xx}, \\ \phi^*(x,0) = \phi_0. \end{cases}$$

But $(\phi, R_0, R_1) = (\sigma\phi^*, \sigma R_0^*, \sigma R_1^*)$, so that

$$\begin{cases} \phi(x,t) = -(M\phi)_x + \frac{1}{2}(V\phi)_{xx}, \\ \phi(x,0) = \sigma\phi_0. \end{cases}$$

Now Lemma 20 implies

$$|R(t)| \le \left[|R(0)| + \|\phi_0\|_{L_1(0,1)} \int_0^t \mathcal{M}_1(s)\,ds \right] + \int_0^t \|\phi_0\|_{L_1(0,1)}\,\mathcal{M}_2(s)|R(s)|\,ds \tag{48}$$

hence there is a constant γ depending only on T and initial data so that $|R(t)| \le \gamma$. Thus the γ that appears in Proposition 12 and in subsequent results can be bounded by the right side of (48) in terms of T and the initial data, which proves the claim.

Existence then follows from Schaefer's fixed point theorem [7, §9.2, Theorem 4]. ■

7 Uniqueness and Stability

Now we shall prove Theorem 2. For notational simplicity, let $\bar{\phi}(x,t) = \phi(x,t) - \phi^*(x,t)$; define $\bar{M}$, $\bar{V}$, and $\bar{R}$ similarly. For any (x,t) we have

$$\bar{M} = M(x,t,R(t)) - M(x,t,R^*(t))$$

so that

$$|\bar{M}| \leq \left|\frac{\partial M}{\partial R}(x,t,\lambda R(t) + (1-\lambda)R^*(t))\right| \; |\bar{R}(t)|$$

for some $0 \leq \lambda \leq 1$. Because $B_1 \hookrightarrow L_1(0,1)$, we see that $R, R^* \in C[0,T]$, so there is a single bounded interval $[a,b]$ so that $\lambda R(t) + (1-\lambda)R^*(t) \in [a,b]$ for all $0 \leq \lambda \leq 1$ and all $0 \leq t \leq T$. Thus, there exists a constant C depending only on $\|R, R^*\|_{C^0[0,1]}$ so that

$$|\bar{M}| \leq C|\bar{R}(t)|$$

for all x.

Using the representations of R and R^*, we find that

$$\begin{aligned}|\bar{M}(x,t)| &\leq C\left|\int_0^t \int_0^1 (M\phi - M^*\phi^*)\,dx\,ds\right| + C|\bar{R}(0)| \\ &\leq C\int_0^t \int_0^1 (|\bar{M}\phi| + |M^*\bar{\phi}|)\,dx\,ds + C|\bar{R}(0)|.\end{aligned}$$

From here, we can apply Gronwall's inequality. Indeed, if

$$\mu(t) = \sup_{0\leq x\leq 1} |\bar{M}|$$

then

$$\mu(t) \leq C\int_0^t \mu(s)\int_0^1 |\phi(y,s)|\,dy\,ds + C\int_0^t \int_0^1 |M^*\bar{\phi}|\,dy\,ds + C|\bar{R}(0)|.$$

Thus the integral form of Gronwall's inequality [3, Thm 2.1] implies

$$\mu(t) \leq \left\{C\int_0^t \int_0^1 |M^*\bar{\phi}|\,dx\,ds + C|\bar{R}(0)|\right\}\exp\left\{C\int_0^t \int_0^1 |\phi(y,s)|\,dy\,ds\right\}$$

and so letting C depend on $\|\phi\|_{C([0,T];B_1)}$ and T, we find

$$|\bar{M}(x,t)| \leq C\int_0^t \int_0^1 |M^*\bar{\phi}|\,dx\,ds + C|\bar{R}(0)|. \tag{49}$$

Now $|\bar{R}(t)| \leq |R(t) - R(0) - R^*(t) + R^*(0)| + |\bar{R}(0)|$ and hence

$$|\bar{R}(t)| \leq \int_0^t \int_0^1 |\bar{M}\phi|\,dy\,ds + \int_0^t \int_0^1 |M^*\bar{\phi}|\,dy\,ds + |\bar{R}(0)|$$

so that (49) implies

$$\begin{aligned}|\bar{R}(t)| &\leq C\int_0^t \int_0^1 |M^*\bar{\phi}|\,dx\,ds + C|\bar{R}(0)| \\ &\leq C\int_0^t \int_0^1 x(1-x)|\bar{\phi}|\,dx\,ds + C|\bar{R}(0)|.\end{aligned} \tag{50}$$

where C depends on $\|R_0^*, R_1^*\|_{C^0[0,T]}$ and $\|\phi^*\|_{C([0,T];B_1)}$.

By subtraction, we see that

$$\bar{\phi}_t = -(M^*\bar{\phi})_x + \tfrac{1}{2}(V^*\bar{\phi})_{xx} - (\bar{M}\phi)_x + \tfrac{1}{2}(\bar{V}\phi)_{xx}.$$

Because $\bar{\phi} \in L_2(0,T;B_2)$, we know that $\bar{\phi}_t \in L_2(0,T;B_0)$ (Corollary 12) and so taking the inner product with $\bar{\phi}$ in B_0 we find for every $0 \le t \le T$ that

$$\begin{aligned}\frac{1}{2}\int_0^1 x(1-x)\bar{\phi}^2\,dx\Big|_t &+ \frac{1}{2}\int_0^t\int_0^1 (V^*\bar{\phi})_x[x(1-x)\bar{\phi}]_x\,dx\,ds\\ &= \frac{1}{2}\int_0^1 x(1-x)\bar{\phi}_0^2\,dx + \int_0^t\int_0^1 M^*\bar{\phi}[x(1-x)\bar{\phi}]_x\,dx\,ds\\ &+ \int_0^t\int_0^1 \bar{M}\phi[x(1-x)\bar{\phi}]_x\,dx\,ds - \frac{1}{2}\int_0^t\int_0^1 (\bar{V}\phi)_x[x(1-x)\bar{\phi}]_x\,dx\,ds.\end{aligned}$$

Young's inequality then implies

$$\begin{aligned}\int_0^1 x(1-x)\bar{\phi}^2\,dx\Big|_t &+ \int_0^t\int_0^1 [x(1-x)\bar{\phi}]_x^2\,dx\,ds\\ &\le C\int_0^1 x(1-x)\bar{\phi}_0^2\,dx + C\int_0^t\int_0^1 x(1-x)\bar{\phi}^2\,dx\,ds\\ &\quad + C\int_0^t\int_0^1 \left\{\bar{m}x(1-x)\phi)^2 + (\bar{v}x(1-x)\phi)_x^2\right\}\,dx\,ds.\end{aligned}$$

To estimate the last term, we first note that

$$|\bar{m}| \le \left|\frac{\partial m}{\partial R}(x,t,\lambda R(t) + (1-\lambda)R^*(t))\right|\,|\bar{R}| \le C|\bar{R}|.$$

Similarly,

$$|\bar{v}| + |\bar{v}_x| \le C\left\{\left|\frac{\partial v}{\partial R}\right| + \left|\frac{\partial^2 v}{\partial R\partial x}\right|\right\}|\bar{R}| \le C|\bar{R}|.$$

Thus, if we use (50) we find

$$\begin{aligned}\int_0^1 x(1-x)\bar{\phi}^2\,dx\Big|_t &+ \int_0^t\int_0^1 [x(1-x)\bar{\phi}]_x^2\,dx\,ds\\ &\le C\int_0^1 x(1-x)\bar{\phi}_0^2\,dx + C\int_0^t\int_0^1 x(1-x)\bar{\phi}^2\,dx\,ds\\ &\quad + C\left\{\int_0^t\int_0^1 x(1-x)|\bar{\phi}|\,dy\,ds + C|\bar{R}(0)|\right\}^2\\ &\quad\quad \cdot\int_0^t\int_0^1 \{x(1-x)\phi^2 + [x(1-x)\phi]_x^2\}\,dx\,ds\end{aligned}$$

and thus

$$\int_0^1 x(1-x)\bar{\phi}^2\,dx\Big|_t + \int_0^t \int_0^1 [x(1-x)\bar{\phi}]_x^2\,dx\,ds$$
$$\leq C\int_0^1 x(1-x)\bar{\phi}_0^2\,dx + C|\bar{R}(0)|^2 + C\int_0^t \int_0^1 x(1-x)\bar{\phi}^2\,dx\,ds.$$

Applying Gronwall's inequality, we find

$$\int_0^1 x(1-x)\bar{\phi}^2\,dx\Big|_t \leq Ce^{Ct}\left(\int_0^1 x(1-x)\bar{\phi}_0^2\,dx + |\bar{R}(0)|^2\right).$$

Thus the definition of $R(t)$ and the embedding $B_1 \hookrightarrow L_1(0,1)$ imply

$$\sup_{0\leq t\leq T}\int_0^1 x(1-x)(\phi-\phi^*)^2\,dx\Big|_t$$
$$+\int_0^T \int_0^1 [x(1-x)(\phi-\phi^*)]_x^2\,dx\,dt$$
$$\leq C\int_0^1 x(1-x)(\phi_0-\phi_0^*)^2\,dx + \int_0^1 [x(1-x)(\phi_0-\phi_0^*)]_x^2$$
$$+ C|R_0(0)-R_1(0)-R_0^*(0)+R_1^*(0)|^2.$$

This then gives us conclusion 1 of Theorem 2. To prove conclusion 2, note that the uniqueness implied by conclusion 1 implies that ϕ and ϕ^* must satisfy the estimates of Theorem 1. In particular, all of the quantities on which C depends have been estimated in terms of initial data. ■

8 Asymptotic Behavior

Now we restrict our attention to the specific choices for M and V made in (5) and (6), namely (after rescalings; see also [16])

$$V = x(1-x), \tag{51a}$$
$$M = \kappa x(1-x)(\rho - R(t)). \tag{51b}$$

Here κ represents selection strength and ρ is the optimal trait mean.

There are two important biological questions in the limit $t \to \infty$,

- What is the behavior of the trait mean $R(t)$ as $t \to \infty$?
- What is the behavior of the total genetic variance $S^2(t)$ as $t \to \infty$? Here

$$S^2(t) = \langle \phi(\cdot,t), 1\rangle_{B_0} = \int_0^1 x(1-x)\phi(x,t)\,dx.$$

We begin with the following relationship between S^2 and R.

Proposition 23 *Let ϕ be the solution of Theorem 1, under the assumptions* (51)*, and let $R_0 = R(0) = R_0(0) + R_1(0) + \int_0^1 (x - 1/2)\phi_0(x)\,dx$. Then*

$$R(t) - \rho = (R_0 - \rho)\exp \int_0^t \int_0^1 -\kappa\, x(1-x)\phi(x,\tau)\,dx\,d\tau$$
$$= (R_0 - \rho)\exp \int_0^t -\kappa\, S^2(\tau)\,d\tau.$$

Proof: From Theorem 1 and (51), for any $0 \le t_1 < t_2 < T$

$$\begin{aligned} R(t_2) - R(t_1) &= \int_{t_1}^{t_2} \int_0^1 M\phi\,dx\,dt \\ &= \int_{t_1}^{t_2} \kappa\,(\rho - R) \int_0^1 x(1-x)\phi\,dx\,dt. \end{aligned} \tag{52}$$

Now $\phi \in C([0,T); B_1)$ so $t \mapsto \int_0^1 x(1-x)\phi(x,t)\,dx$ is continuous; indeed

$$\begin{aligned} &\left| \int_0^1 x(1-x)[\phi(x,t_2) - \phi(x,t_1)]\,dx \right| \\ &\quad \le \left(\int_0^1 x(1-x)\,dx \right)^{\frac{1}{2}} \left(\int_0^1 x(1-x)[\phi(x,t_2) - \phi(x,t_1)]^2\,dx \right)^{\frac{1}{2}} \\ &\qquad \le \frac{1}{6}\,\|\phi(\cdot,t_2) - \phi(\cdot,t_1)\|_{B_0} \le \frac{1}{6}\,\|\phi(\cdot,t_2) - \phi(\cdot,t_1)\|_{B_1}. \end{aligned}$$

Dividing (52) by $t_2 - t_1$ and letting $t_2 \to t_1$, we find that

$$R'(t) = \kappa\,(\rho - R) \int_0^1 x(1-x)\phi(x,t)\,dx$$

for all $0 < t < T$. This gives us the linear equation for $(R - \rho)$

$$(R-\rho)' + \kappa\,(R - \rho) \int_0^1 x(1-x)\phi(x,t)\,dx = 0$$

which is equivalent to

$$\frac{d}{dt}\left\{ (R-\rho)\exp \int_0^t \int_0^1 \kappa\, x(1-x)\phi(x,\tau)\,dx\,d\tau \right\} = 0$$

and has solution

$$R(t) - \rho = (R_0 - \rho)\exp \int_0^t \int_0^1 -\kappa\, x(1-x)\phi(x,\tau)\,dx\,d\tau.$$

■.

Now we turn our attention to the total genetic variance.

Proposition 24 *Let ϕ be the solution of Theorem 1 under the assumptions* (51). *Then*

$$\int_0^\infty \int_0^1 x(1-x)\phi(x,t)\,dx\,dt < \infty.$$

Remark: This is a weak way of saying that $S^2(t) \to 0$ as $t \to \infty$.

Remark: Although Theorem 1 only guarantees the existence of a solution on $[0, T)$ for finite $T < \infty$, the fact that T is arbitrary and that solutions are unique 2 lets us define $\phi(x,t)$ for all $0 \leq t < \infty$.

Proof: Let $0 < T < \infty$ be arbitary; then the equation implies

$$\phi_t = -(M\phi)_x + \tfrac{1}{2}(V\phi)_{xx}$$

in $L_2(0,T;B_0)$. Then, taking the inner product with 1 in B_0 and using the particular forms for M and V from (51), we find

$$\int_0^1 x(1-x)\phi_t\,dx = -\int_0^1 x(1-x)[\kappa\, x(1-x)(\rho-R)\phi]_x\,dx \\ + \frac{1}{2}\int_0^1 x(1-x)[x(1-x)\phi]_{xx}\,dx$$

in $L_2(0,T)$. Because $C^\infty[0,1]$ is dense in B_2, we can integrate by parts to see that

$$\frac{1}{2}\int_0^1 x(1-x)[x(1-x)\phi]_{xx}\,dx = -\frac{1}{2}\int_0^1 [x(1-x)]_x[x(1-x)\phi]_x\,dx$$

and because $x(1-x)\phi(\cdot,t) \in B_1 \hookrightarrow \overset{\circ}{W}{}^1_2(0,1)$ we have

$$\frac{1}{2}\int_0^1 x(1-x)[x(1-x)\phi]_{xx}\,dx = \frac{1}{2}\int_0^1 [x(1-x)]_{xx}x(1-x)\phi\,dx.$$

Similarly

$$\begin{aligned} -\int_0^1 & x(1-x)[\kappa\, x(1-x)(\rho-R)\phi]_x\,dx \\ &= -\kappa\,(\rho-R)\int_0^1 x(1-x)[x(1-x)\phi]_x\,dx \\ &= -\kappa\,(R-\rho)\int_0^1 (2x-1)x(1-x)\phi\,dx. \end{aligned}$$

Thus

$$\frac{d}{dt}\int_0^1 x(1-x)\phi\,dx = \kappa\,(R-\rho)\int_0^1 (2x-1)x(1-x)\phi\,dx \\ -\int_0^1 x(1-x)\phi\,dx.$$

We can now substitute for $R - \rho$ using Proposition 23 to find

$$\begin{aligned}\frac{d}{dt}\int_0^1 x(1-x)\phi\,dx = &-\int_0^1 x(1-x)\phi\,dx \\ &+\kappa(R_0-\rho)\left\{\exp\int_0^t\int_0^1 -\kappa\,x(1-x)\phi(x,\tau)\,dx\,d\tau\right\} \\ &\cdot\int_0^1 (2x-1)x(1-x)\phi\,dx\end{aligned}$$

as elements of $L_2(0,T)$. However, the continuity $\phi \in C([0,T);B_1)$ implies that the right side and hence the left side are continuous; thus it holds for all time $0 < t < \infty$.

Let $0 < \delta < 1$ be chosen arbitrarily. Because $\phi \geq 0$ and because $\kappa > 0$, we know that the function

$$t \mapsto \exp\int_0^t\int_0^1 -\kappa\,x(1-x)\phi(x,\tau)\,dx\,d\tau$$

is monotone non-increasing. Then, either for every t we have

$$\kappa|R_0-\rho|\left\{\exp\int_0^t\int_0^1 -\kappa\,x(1-x)\phi(x,\tau)\,dx\,d\tau\right\} > \delta$$

or there exists some $T(\delta)$ so that

$$\kappa|R_0-\rho|\left\{\exp\int_0^t\int_0^1 -\kappa\,x(1-x)\phi(x,\tau)\,dx\,d\tau\right\} \leq \delta$$

for all $t \geq T(\delta)$.

If the first case obtains, then

$$\int_0^t\int_0^1 x(1-x)\phi(x,\tau)\,dx\,d\tau \leq \frac{1}{\kappa}\ln\left(\frac{\kappa|R_0-\rho|}{\delta}\right)$$

for all t, which would prove our result.

On the other hand, in the second case, we have

$$\frac{d}{dt}\int_0^1 x(1-x)\phi\,dx \leq -\int_0^1 x(1-x)\phi\,dx + \delta\left|\int_0^1 (2x-1)x(1-x)\phi\,dx\right|$$

for all $t \geq T(\delta)$. Then, because $\phi \geq 0$,

$$\frac{d}{dt}\int_0^1 x(1-x)\phi\,dx \leq -(1-\delta)\int_0^1 x(1-x)\phi\,dx.$$

Solving the differential inequality implies

$$\int_0^1 x(1-x)\phi(x,t)\,dx \leq \left(\int_0^1 x(1-x)\phi(x,T(\delta))\,dx\right)e^{-(1-\delta)t}$$

for all $t \geq T(\delta)$. Because $\phi \in C([0,T(\delta)+1),B_1)$, this is sufficient to imply our result. ■.

Corollary 25 *Let ϕ be the solution of Theorem 1 under the assumptions* (51). *Suppose that there is a constant $0 < \delta < 1$ so that*

$$|R_0 - \rho| \leq \delta/\kappa.$$

Then, for any $t > 0$

$$\int_0^1 x(1-x)\phi(x,t)\,dx \leq \left(\int_0^1 x(1-x)\phi_0(x)\,dx\right) e^{-(1-\delta)t}.$$

Proof: Because $\phi \geq 0$, our hypotheses are sufficient to guarantee that the second case of the previous proof obtains, with $T(\delta) = 0$. ■

Corollary 26 *Let ϕ be the solution of Theorem 1 under the assumptions* (51). *Suppose that there is a constant $0 < \delta < 1$ so that*

$$|R_0 - \rho| \leq \delta/\kappa.$$

Then, for any $t > 0$

$$|R(t) - \rho| \\ \geq |R_0 - \rho| \exp\left\{\frac{\kappa}{1-\delta}\left(\int_0^1 x(1-x)\phi_0(x)\,dx\right)\left(e^{-(1-\delta)t} - 1\right)\right\}.$$

Proof: From Proposition 23

$$R(t) - \rho = (R_0 - \rho)\exp\left[\int_0^t \int_0^1 -\kappa\, x(1-x)\phi(x,\tau)\,dx\,d\tau\right].$$

Then, because

$$\int_0^1 x(1-x)\phi(x,t)\,dx \leq \left(\int_0^1 x(1-x)\phi_0(x)\,dx\right) e^{-(1-\delta)t},$$

we have

$$|R(t) - \rho| \geq |R_0 - \rho| \exp\left[-\kappa\left(\int_0^1 x(1-x)\phi_0(x)\,dx\right)\int_0^t e^{-(1-\delta)t}\,d\tau\right].$$

Evaluating the inner integral gives us the result. ■

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References

[1] M. Abramozitz and I. A. Stegun, *Handbook of mathematical functions*, Dover Publications, 1965.

[2] M. Campiti, G. Metafune and D. Pallara, *Degenerate self-adjoint evolution equations on the unit interval*, Semigroup Forum 57 (1998), 1–36.

[3] C. Chicone, *Ordinary Differential Equations with Applications*, Springer-Verlag, New York, 1999

[4] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw Hill, New York, 1955.

[5] J. F. Crow and M. Kimura, *An introduction to population genetics theory*, Harper & Row, New York, 1970.

[6] E. DiBenedetto, *Real analysis*, Birkhäuser, Boston, 2002.

[7] L. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[8] W.J. Ewens, *Mathematical population genetics*, Springer, Berlin, 1979.

[9] A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, *Degenerate second order differential operators generating analytic semigroups in L^p and $W^{1,p}$*, Math. Nachrichten 238 (2002), 78–102.

[10] R. Ghez, *Diffusion phenomena*, Kluwer, New York, 2001.

[11] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 1984.

[12] W. G. Hill and J. Rasbash, *Models of long term artificial selection in finite population*, Genet. Res., Camb. 48 (1986), 41–50.

[13] A. V. Ivanov, *boundary-value problem for degenerate second-order linear parabolic equations*, in O. A. Ladyzhenskaya, ed., *Boundary value problems of mathematical physics and related aspects of function theory: part IV*, Consultants Bureau, New York, 1971, 22–43.

[14] M. Kimura, *Diffusion models in population genetics*, J. Applied Probability 1 (1964), 177–232.

[15] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Trans. AMS, vol. 23, American Mathematical Society, Providence, RI, 1968.

[16] J. R. Miller, M. C. Pugh, and M. B. Hamilton, *A finite locus effect diffusion model for the evolution of a quantitative trait*, in review.

[17] O. A. Oleinik and E. V. Radkevic, *Second order equations with nonnegative characteristic form*, American Mathematical Society, Providence, 1973.

[18] W. Paul and J. Baschnagel, *Stochastic processes: from physics to finance*, Springer-Verlag, Berlin, 1990.

[19] T. Shiga, *Existence and uniqueness of solutions for a class of non-linear diffusion equations*, J. Math. Kyoto Univ. 27 (1987), 195–215.

[20] N. Shimakura, *Existence and uniqueness of solutions for a diffusion model of intergroup selection*, J. Math. Kyoto Univ. 25 (1985), 775–788.